Approximating Language Edit Distance Beyond Fast Matrix Multiplication: Ultra-linear Grammars & Hardness of Parsing

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Abstract

In 1975, a breakthrough result of L. Valiant showed that parsing context free grammars can be reduced to Boolean matrix multiplication, resulting in a running time of $O(n^\omega)$ for parsing where $\omega \leq 2.373$ is the exponent of fast matrix multiplication, and n is the string length. Recently, Abboud, Backurs and V. Williams (FOCS 2015) demonstrated that this is likely optimal; moreover, a combinatorial $o(n^3)$ algorithm is unlikely to exist for the general parsing problem[1]. The language edit distance problem is a significant generalization of the parsing problem, which computes the minimum edit distance of a given string (using insertions, deletions, and substitutions) to any valid string in the language, and has received significant attention both in theory and practice since the seminal work of Aho and Peterson in 1972. Clearly, the lower bound for parsing rules out any algorithm running in $o(n^\omega)$ time that can return a nontrivial multiplicative approximation of the language edit distance problem. Furthermore, combinatorial algorithms with cubic running time or algorithms that use fast matrix multiplication are often not desirable in practice.

To break this n^ω hardness barrier, in this paper we study *additive* approximation algorithms for language edit distance. We provide two explicit combinatorial algorithms to obtain a string with minimum edit distance with performance dependencies on either the number of *non-linear productions*, k^*, or the number of *nested non-linear production*, k, used in the optimal derivation. Explicitly, we give an additive $O(k^*\gamma)$ approximation in time $O(|G|(n^2 + \frac{n^3}{\gamma^3}))$ and an additive $O(k\gamma)$ approximation in time $O(|G|(n^2 + \frac{n^3}{\gamma^2}))$, where $|G|$ is the grammar size and n is the string length. In particular, we obtain tight approximations for an important subclass of context free grammars known as *ultralinear* grammars, for which k and k^* are naturally bounded. Interestingly, we show that while for notable subclasses of ultralinear grammars, we can design $O(|G|n^2)$ algorithms, the same conditional lower bound for parsing context free grammars holds for the class of ultralinear grammars, thus clearly demarking where parsing become hards!

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[1] with any polynomial dependency on the grammar size

1 Introduction

Introduced by Chomsky in 1956 [Cho59], context-free grammars (CFG) play a fundamental role in the development of formal language theory [1, 24], compiler optimization [17, 46], natural language processing [30, 35], with diverse applications in areas such as computational biology [38, 8], machine learning [21, 32, 4] and databases [26, 15, 39]. Parsing CFG is a basic computer science question, that given a CFG G over an alphabet Σ, and a string $x \in \Sigma^*$, $|x| = n$, determines if x belongs to the language $\mathcal{L}(G)$ generated by G. The canonical parsing algorithms such as Cocke-Younger-Kasimi (CYK) [1], Earley parser, [13] etc. are based on a natural dynamic programming, and run in $O(n^3)$ time[2]. In 1975, in a theoretical breakthrough, Valiant gave a reduction from parsing to Boolean matrix multiplication, showing that the parsing problem can be solved in $O(n^\omega)$ time [42]. Despite decades of efforts, these running times have remain completely unchanged.

Nearly three decades after Valiant's result, Lee came up with an ingenious reduction from Boolean matrix multiplication to CFG parsing, showing for the first time why known parsing algorithms may be optimal [28]. A remarkable recent result of Abboud, Backurs and V. Williams made her claims concrete [3]. Based on a conjecture of the hardness of computing large cliques in graphs, they ruled out any improvement beyond Valiant's algorithm; moreover they showed that there can be no combinatorial algorithm for CFG parsing that runs in truly subcubic $O(n^{3-\epsilon})$ time for $\epsilon > 0$ [3]. However combinatorial algorithms with cubic running time or algorithms that use fast matrix multiplication are often impractical. Therefore, a long-line of research in the parsing community has been to discover subclasses of context free grammars that are sufficiently expressive yet admit efficient parsing time [29, 27, 18]. Unfortunately, there still exist important subclasses of the CFG's for which neither better parsing algorithms are known, nor have conditional lower bounds been proven to rule out the possibility of such algorithms.

Language Edit Distance.

A generalization of CFG parsing, introduced by Aho and Peterson in 1972 [2], is *language edit distance* (LED) which can be defined as follows.

▶ **Definition 1** (Language Edit Distance (LED))**.** Given a formal language $\mathcal{L}(G)$ generated by a grammar G over alphabet Σ, and a string $\overline{x} \in \Sigma^*$, compute the minimum number of edits (insertion, deletion and substitution) needed on $\overline{x}$ to convert it to a valid string in $\mathcal{L}(G)$.

LED is among the most fundamental and best studied problems related to strings and grammars [2, 34, 39, 40, 10, 3, 36, 7, 23], and generalizes two basic problems in computer science: parsing and string edit distance computation. Aho and Peterson presented a dynamic programming algorithm for LED that runs in $O(|G|^2 n^3)$ time [2], which was improved to $O(|G|n^3)$ by Myers in 1985 [34]. Only recently these bounds have been improved by Bringmann, Grandoni, Saha, and V. Williams to give the first truly subcubic $O(n^{2.8244})$ algorithm for LED [10]. When considering approximate answers, a *multiplicative* $(1+\epsilon)$-approximation for LED has been presented by Saha in [40], that runs in $O(\frac{n^\omega}{\text{poly}(\epsilon)})$ time.

These subcubic algorithms for LED crucially use fast matrix multiplication, and hence are not practical. Due to the hardness of parsing [28, 3], LED cannot be approximated

[2] dependency on the grammar size if not specified is either $|G|$ as in most combinatorial algorithms, or $|G|^2$ as in most algebraic algorithms.

with any multiplicative factor in time $o(n^\omega)$. Moreover, there cannot be any combinatorial multiplicative approximation algorithm that runs in $O(n^{3-\epsilon})$ time for any $\epsilon > 0$ [3]. LED provides a very generic framework for modeling problems with vast applications [26, 22, 44, 31, 37, 35, 16]. A fast exact or approximate algorithm for it is likely to have tangible impact, yet there seems to be a bottleneck in improving the running time beyond $O(n^\omega)$, or even in designing a truly subcubic combinatorial approximation algorithm. Can we break this n^ω barrier?

One possible approach is to allow for an *additive approximation*. Since the hardness of multiplicative approximation arise from the lower bound of parsing, it is possible to break the n^ω barrier by designing a purely combinatorial algorithm for LED with an additive approximation. Such a result will have immense theoretical and practical significance. Due to the close connection of LED with matrix products, all-pairs shortest paths and other graph algorithms [40, 10], this may imply new algorithms for many other fundamental problems. In this paper, we make a significant progress in this direction by providing the first nontrivial additive approximation for LED that runs in quadratic time. Let $G = (Q, \Sigma, P, S)$ denote a context free grammar, where Q is the set of nonterminals, Σ is the alphabet or set of terminals, P is the set of productions, and S is the starting non-terminal.

▶ **Definition 2.** Given $G = (Q, \Sigma, P, S)$, a production $A \to \alpha$ is said to be *linear* if there is at most one non-terminal in α where $A \in Q$ and $\alpha \in (Q \cup \Sigma)^*$. Otherwise, if α contains two or more non-terminals, then $A \to \alpha$ is said to be *non-linear*

The performance of our algorithms depends on either the total number of *non-linear productions* or the maximum number of *nested non-linear productions* (depth of the parse tree after condensing every consecutive sequence of linear productions, more formally defined in Appendix 5.2) in the derivation of string with optimal edit distance, where the latter is often substantially smaller. Explicitly, we give an additive $O(k^*\gamma)$ approximation in time $O(|G|(n^2 + \frac{n^3}{\gamma^3}))$ and an additive $O(k\gamma)$ approximation in time $O(|G|(n^2 + \frac{n^3}{\gamma^2}))$, where k^* is the number of non-linear productions in the derivation of the optimal string, and k is the maximum number of nested non-linear productions in the derivation of the optimal string (each minimized over all possible derivations).

Our algorithms will be particularly useful for an important subclass of CFGs, known as the *ultralinear grammars*, for which these values are tightly bounded for all derivations [48, 12, 29, 9, 33].

▶ **Definition 3** (ultralinear)**.** A grammar $G = (Q, \Sigma, P, S)$ is said to be **k-ultralinear** if there is a partition $Q = Q_1 \cup Q_1 \cup \cdots \cup Q_k$ such that for every $X \in Q_i$, the productions of X consist of *linear productions* $X \to \alpha A | A\alpha | \alpha$ for $A \in Q_j$ with $j \leq i$ and $\alpha \in \Sigma$, or *non-linear productions* of the form $X \to w$, where $w \in (Q_1 \cup Q_2 \cup \cdots \cup Q_{i-1})^*$.

The parameter k places a built-in upper bound on the number of nested non-linear productions allowed in any derivation. Thus for simplicity we will use k both to refer to the parameter of an ultralinear grammar, as well as the maximum number of nested non-linear productions. Furthermore, if d is the maximum number of non-terminals on the RHS of a production, then d^k is a built-in upper bound on the total number of non-linear productions in any derivation. In all our algorithms, without loss of generality, we use a standard normal form where $d = 2$ for all non-linear productions. As we will see later, given any CFG G and any $k \geq 1$, we can create a new grammar G' by making k copies $Q_1, \ldots, Q_k$ of the set of non-terminals Q of G, and forcing every nonlinear production in Q_i to go to non-terminals in Q_{i-1}. Thus G' has non-terminal set $Q_1 \cup Q_2 \cup \cdots \cup Q_k$, and size $O(k|G|)$. In this way

we can restrict any CFG to a k-ultralinear grammar which can produce any string in $\mathcal{L}(G)$ requiring no more than k nested non-linear productions. It is precisely this procedure of creating a k-ultralinear grammar from a CFG G that we use in our proof of hardness for parsing ultralinear languages (Theorem 24).

For example, if G is the well-known *Dyck Languages* [39, 7], the language of well-balanced parenthesis, $\mathcal{L}(G')$ contains the set of all parentheses strings with at most k-levels of nesting. Note that a string consisting of n open parenthesis followed by n matching closed parenthesis has zero levels of nesting, whereas the string "((())())" has one level. As an another example, consider RNA-folding [10, 43, 49] which is a basic problem in computational biology and can be modeled by grammars. The restricted language $\mathcal{L}(G')$ for RNA-folding denotes the set of all RNA strings with at most k-nested folds. In typical applications, we do not expect the number of nested non-linear productions used in the derivation of a valid string to be too large [15, 26, 5].

Among our other results, we consider exact algorithms for several other notable sub-classes of the CFG's. In particular, we develop exact quadratic time language edit distance algorithms for the linear, metalinear, and superlinear languages. Moreover, we show matching lower bound assuming the Strong Exponential Time Hypothesis [19, 20]. The figure to the right displays the hierarchical relationship between these grammars, where all upwards lines denote strict containment. Interestingly, till date there exists no parsing algorithm for the ultralinear grammars that run in time $o(n^\omega)$, while $O(n^2)$ algorithm exists for the metalinear grammars. In addition, there is no combinatorial algorithm that runs in $o(n^3)$ time. In this paper, we derive conditional lower bound exhibiting why a faster algorithm has so far been elusive for the ultralinear grammars, clearly demarking the boundary where parsing becomes hard!

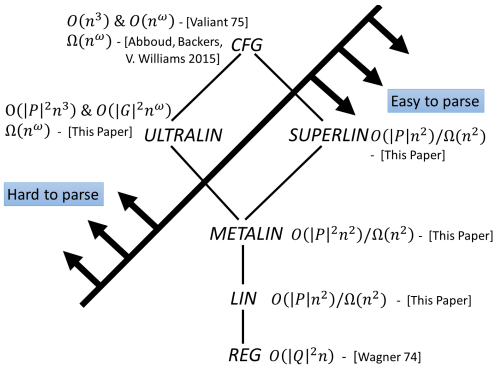

Figure 1 CFG Hierarchy & Our Results

1.1 Results & Techniques

Lower Bounds. Our first hardness result is a lower bound for the problem of linear language edit distance. We show that a truly subquadratic time algorithm for linear language edit distance would refute the Strong Exponential Time Hypothesis (SETH). This further builds on a growing family of "SETH-hard" problems – those for which lower bounds can be proven conditioned on SETH. We prove this result by reducing binary string edit distance, which has been shown to be SETH-hard [11, 6], to linear language edit distance.

▶ Theorem (23). There exists no algorithm to compute the minimum edit distance between a string $\overline{x}$, $|\overline{x}| = n$, and a linear language $\mathcal{L}(G)$ in $o(n^{2-\epsilon})$ time for any constant $\epsilon > 0$, unless SETH is false.

Our second, and primary hardness contribution is a conditional lower bound on the recognition problem for ulralinear languages. Our result builds closely off of the work of Abboud, Backurs and V. Williams [3], who demonstrate that finding an $o(n^3)$-time combinatorial algorithm or any $o(n^\omega)$-algorithm for context free language recognition would result in faster algorithms for the k-clique problem and falsify a well-known conjecture in

graph algorithms. We modify the grammar in their construction to be ultralinear, then demonstrate that the same hardness result for our grammar. See Appendix 5.3 for details.

▶ Theorem (24). There is a ultralinear grammar $\mathcal{G}_U^\ell = \mathcal{G}_U$ such that if we can solve the membership problem for a string of length n in time $O(|\mathcal{G}_U|^\alpha n^c)$ for any fixed constant $\alpha > 0$, then we can solve the k-clique problem on a graph with n nodes in time $O(n^{c(k+3)+3\alpha})$.

Upper Bounds. We provide the first quadratic time algorithms for linear (Theorem 7), superlinear (Theorem 28), and metalinear language edit distance (Theorem 29), running in $O(|P|n^2)$, $O(|P|n^2)$ and $O(|P|^2n^2)$ time respectively. This exhibits a large family of grammars for which edit distance computation can be done faster than for general context free grammars, as well as for other well known grammars such as the Dyck grammar [3]. Along with our lower bound for the ultralinear language parsing, this demonstrates a clear division between those grammars for which edit distance can be efficiently calculated, and those for which the problem is likely to be fundamentally hard. Our algorithms build progressively off the construction of a *linear language edit distance graph*, reducing the problem of edit distance computation to computing shortest path on a graph with $O(|P|n^2)$ edges (Section 2).

Our main contribution is an additive approximation for language edit distance. We first present a cubic time exact algorithm, and then show a general procedure for modifying this algorithm, equivalent to forgetting states of the underlying dynamic programming table, into a family of *amnesic* dynamic programming algorithms. This produces *additive approximations* of the edit distance, and also provides a tool for proving general bounds on any such algorithm. In particular, we provide two explicit procedures for forgetting dynamic programming states: *uniform* and *non-uniform* grid approximations achieving the following approximation-running time trade-off. See Section 4, and Appendix 5.1 for missing proofs.

▶ **Theorem 4.** *If $\mathcal{A}$ is a γ-uniform grid approximation, then the edit distance computed by $\mathcal{A}$ satisfies $|OPT| \leq |\mathcal{A}| \leq |OPT| + O(k^*\gamma)$ and it runs in in $O(|P|(n^2 + (\frac{n}{\gamma})^3))$ time.*

▶ **Theorem 5.** *Let $\mathcal{A}$ be any γ-non-uniform grid approximation, then the edit distance computed by $\mathcal{A}$ satisfies $|OPT| \leq |\mathcal{A}| \leq |OPT| + O(k\gamma)$and it runs in $O\big(|P|\big(n^2 + \frac{n^3}{\gamma^2}\big)\big)$ time.*

We believe that our amnesic technique can be applied to wide range of potential dynamic programming approximation algorithms, and lends itself particularly well to randomization.

2 Linear Grammar Edit Distance in Quadratic Time

We first introduce a graph-based exact algorithm for linear grammar, that is a grammar $G = (Q, \Sigma, P, S)$ where every production is of the form $A \to \alpha B$ or, $A \to B\alpha$, or $A \to \alpha$ for $A, B \in Q$, and $\alpha \in \Sigma$. Given G and a string $\overline{x} = x_1x_2 \dots x_n \in \Sigma^*$, we give an $O(n^2|P|)$ algorithm to compute edit distance between $\overline{x}$ and G in this section. The algorithm serves as a building block for the rest of the paper.

Note that if we can only have productions of the form $A \to \alpha B$ (or $A \to B\alpha$ but not both) then the corresponding language is regular, and all regular languages can be generated in this manner. However, there are linear languages that are not regular. Therefore, regular languages are a strict subclass of linear languages. Being a natural extension of the regular languages, the properties and applications of linear languages are of much interest[14, 41].

Algorithm. Given inputs G and $\overline{x}$, we construct a weighted digraph $\mathcal{T} = \mathcal{T}(G, \overline{x})$ with a designated vertex $S^{1,n}$ as the source and t as the sink such that the weight of the shortest path between them will be the minimum language edit distance of $\overline{x}$ to G.

Construction. The vertices of $\mathcal{T}$ consist of $\binom{n}{2}$ *clouds*, each corresponding to a unique substring of $\overline{x}$. We use the notation (i, j) to represent the cloud, $1 \leq i \leq j \leq n$, corresponding to

the substring $x_i x_{i+1} .... x_j$. Each cloud will contain a vertex for every nonterminal in Q. Label the nonterminals $Q = \{S = A_1, A_2, \ldots, A_q\}$ where $|Q| = q$, then we denote the vertex corresponding to A_k in cloud (i, j) by $A_k^{i,j}$. We will add a new sink node t, and use $S^{1,n}$ as the source node s. Thus the vertex set of $\mathcal{T}$ is $V(\mathcal{T}) = \{A_k^{i,j} \mid 1 \leq i \leq j \leq n,\ 1 \leq k \leq q\} \cup \{t\}$. The edges of $\mathcal{T}$ will correspond to the productions in G. Each path from a nonterminal $A_k^{i,j}$ in (i, j) to t corresponds to the production of a legal string w, that is a string that can be derived starting from A_k and following the productions of P, and a sequence of editing procedures to edit w to $x_i x_{i+1} \ldots x_j$. For any cloud (i, j), edges will exist between two nonterminals in (i, j), and from nonterminals in (i, j) to nonterminals in $(i + 1, j)$ and $(i, j - 1)$. Our goal will be to find the shortest path from $S^{1,n}$, the starting nonterminal S in cloud $(1, n)$, to the sink t.

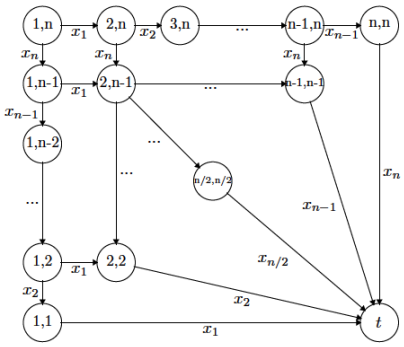

Figure 2 Clouds corresponding to Linear Grammar Edit Distance Graph Construction. Each cloud contains a vertex for every nonterminal

Adding the edges. Each edge in $\mathcal{T}$ is directed, has a weight in $\mathbb{Z}^+$ and a label from $\{x_1, x_2, .., x_n, \epsilon\} \cup \{\epsilon(\alpha) \mid \alpha \in \Sigma\}$. If u, v are two vertices in $\mathcal{T}$, then we use the notation $u \xrightarrow[w(u,v)]{\ell} v$ to denote the existence of an edge from u to v with weight $w(u, v)$ and edge label ℓ. For any nonterminal $A \in Q$, define $null(A)$ to be the length of the shortest string in Σ^* derivable from A, which can be precomputed in $O(|Q||P| \log(|Q|))$ time for all A (see Theorem 30). This is the minimum cost of deleting a whole string produced by A. Given input $x_1 x_2 \ldots x_n$, for all nonterminals A_k, A_t and every $1 \leq i \leq j \leq n$, the construction is as follows:

- **Legal Productions:** For $i \neq j$, then if $A_k \to x_i A_t$ is a production, add the edge $A_k^{i,j} \xrightarrow[0]{x_i} A_t^{i+1,j}$ to $\mathcal{T}$. If $A_k \to A_t x_j$ is a production, add the edge $A_k^{i,j} \xrightarrow[0]{x_j} A_t^{i,j-1}$ to $\mathcal{T}$.
- **Completing Productions:** If $A_k \to x_i$ is a production, add the edge $A_k^{i,i} \xrightarrow[0]{x_i} t$ to $\mathcal{T}$. If $A_k \to x_i A_t$ or $A_k \to A_t x_i$ is a production, add the edge $A_k^{i,i} \xrightarrow[null(A_t)]{x_i} t$ to $\mathcal{T}$.
- **Insertion:** If $A_k \to x_i A_k$ is *not* a production, add the edge $A_k^{i,j} \xrightarrow[1]{x_i} A_k^{i+1,j}$ to $\mathcal{T}$. If $A_k \to A_k x_j$ is *not* a production, add $A_k^{i,j} \xrightarrow[1]{x_j} A_k^{i,j-1}$. *{these are called insertion edges.}*
- **Deletion:** For every production $A_k \to \alpha A_t$ or $A_k \to A_t \alpha$, add the edge $A_k^{i,j} \xrightarrow[1]{\epsilon(\alpha)} A_t^{i,j}$. *{these are called deletion edges.}*
- **Replacement:** For every production $A_k \to \alpha A_t$, if $\alpha \neq x_i$, then add the edge $A_k^{i,j} \xrightarrow[1]{x_i} A_t^{i+1,j}$ to T. For every production $A_k \to A_t \alpha$, if $\alpha \neq x_j$, add $A_k^{i,j} \xrightarrow[1]{x_j} A_t^{i,j-1}$ to $\mathcal{T}$. For any A_k such that $A_k \to x_i$ is not a production, but $A_k \to \alpha$ is a production with $\alpha \in \Sigma$, add the edge $A_k^{i,i} \xrightarrow[1]{x_i} t$ to $\mathcal{T}$.*{these are called substitution or replacement edges.}*

▶ **Theorem 6.** *For every $A_k \in Q$ and every $1 \leq i \leq j \leq n$, the cost of the shortest path of from $A_k^{i,j}$ to the sink $t \in \mathcal{T}$ is d if and only if d is the minimum edit distance between the string $x_i \ldots x_j$ and the set of strings which can be derived from A_k.*

▶ **Theorem 7.** *The cost of the shortest path from $S^{1,n}$ to t in the graph $\mathcal{T}$ is the minimum edit distance which can be computed in $O(|P|n^2)$ time.*

3 Context Free Language Edit Distance

In this section, we develop an exact algorithm which utilizes the graph construction presented in Section 2 to compute the language edit distance of a string $\overline{x} = x_1 \dots x_n$ to any context free grammar (CFG) $G = (Q, \Sigma, P, S)$. We use a standard normal form for G, which is precisely Chomsky normal form except we allow productions of the form $A \to Aa|aA$, for $A \in Q, a \in \Sigma$. Any CFG can be reduced to have such a normal form (see Appendix 5.6).

Let $P_L, P_{NL} \subset P$ be the subsets of (legal) linear and non-linear productions respectively. Then for any nonterminal $A \in Q$, the grammar $G_L = (Q, \Sigma, P_L, A)$ is linear, and we denote the corresponding linear language edit distance graph $\mathcal{T}(G_L, \overline{x}) = \mathcal{T}$, as constructed in Section 2. Let L_i be the set of clouds in $\mathcal{T}$ which correspond to substrings of length i. In other words: $L_i = \{(k, j) \in \mathcal{T} \mid j - k + 1 = i\}$. Then $L_1, \dots, L_n$ is a *layered partition* of $\mathcal{T}$. Let t be the sink of $\mathcal{T}$.

We write $\mathcal{T}^R$ to denote the graph $\mathcal{T}$ where the direction of each edge is reversed. Let L_i^R denote the edge reversed subgraph of L_i. In other words, L_i^R is the subgraph of $\mathcal{T}^R$ with the same vertex set as L_i. Our algorithm will add some additional edges within L_i^R, and some additional edges from t to L_i^R, for all $1 \leq i \leq n$, resulting in an augmented subgraph which we denote $\overline{L}_i^R$. We then compute single source shortest path from t to $\overline{L}_i^R \cup \{t\}$ in phase i. Our algorithm will maintain the property that, after phase $q - p + 1$, if $A^{p,q}$ is any nonterminal in cloud (p, q) then the weight of the shortest path from t to $A^{p,q}$ is precisely the minimum edit distance between the string $x_p x_{p+1} \dots x_q$ and the set of strings that are legally derivable from A. The algorithm is as follows:

Algorithm: Context Free-Exact

1. **Base Case: strings of length 1.** For every non-linear production $A \to BC$, and every $1 \leq \ell \leq n$, add the edges $A^{\ell,\ell} \xleftarrow[null(B)]{} C^{\ell,\ell}$ and $A^{\ell,\ell} \xleftarrow[null(C)]{} B^{\ell,\ell}$ to L_1^R. Note that the direction of the edges are reversed because we are adding edges to L_1^R and not L_1. Call the resulting augmented graph $\overline{L_1^R}$.
2. Solve single source shortest path from t to every vertex in $\overline{L_1^R} \cup \{t\}$. Store the value of the shortest path from t to every vertex in $\overline{L_1^R}$, and an encoding of the path itself. For any $1 \leq p \leq q \leq n$ and $A^{p,q} \in L_{q-p+1}$, we write $T_{p,q}(A)$ to denote the weight of the shortest path from t to $A^{p,q}$. After having computed shortest paths from t to every vertex in the subgraphs $\overline{L}_1^R, \dots, \overline{L}_{i-1}^R$, we now consider L_i^R.
3. **Induction: strings of length i.** For every edge from a vertex $A^{p,q}$ in L_i to a vertex $B^{p+1,q}$ or $B^{p,q-1}$ in L_{i-1} with cost $\gamma \in \{0, 1\}$, add an edge from t to $A^{p,q} \in L_i^R$ with cost $T_{p+1,q}(B) + \gamma$ or $T_{p,q-1}(B) + \gamma$, respectively. These are the linear production edges created in the linear grammar edit distance algorithm.
4. For every non-linear production $A \to BC$ and every vertex $A^{p,q} \in L_i^R$, add an edge from t to $A^{p,q}$ in L_i^R with cost c where $c = \min_{p \leq \ell < q} T_{p,\ell}(B) + T_{\ell+1,q}(C)$. Additionally, to later recover the derivation, we store the specific ℓ which yields the minimum value of the above equation.
5. For every non-linear production $A \to BC$, add the edge $A^{p,q} \xleftarrow[null(B)]{} C^{p,q}$ and $A^{p,q} \xleftarrow[null(C)]{} B^{p,q}$ to L_i^R.
6. After adding the edges in steps 3-5, we call the resulting graph $\overline{L}_i^R$. Then compute shortest path from t to every vertex in the subgraph $\overline{L}_i^R \cup \{t\}$, and store the values of the shortest paths, along with an encoding of the paths themselves.
7. Repeat for $i = 1, 2, \dots, n$. Return the value $T_{1,n}(S)$.

▶ **Theorem 8.** *For any nonterminal $A \in Q$ and $1 \leq p \leq q \leq n$, the weight of the shortest path from $A^{p,q} \in \overline{L}_i$ to t is the minimum edit distance between the substring $x_p \dots x_q$ and the set of strings which can be legally produced from A, and the overall time required to compute the language edit distance is $O(|P|n^3)$.*

4 Context Free Language Edit Distance Approximation

Now this cubic time algorithm itself is not an improvement on that of Aho and Peterson [2]. However, by strategically modifying the construction of the subgraphs L_i by **forgetting** to compute some of the non-linear edge weights, we can obtain an additive approximation of the minimum edit distance. We introduce a family of approximation algorithms which do just this, and prove a strong general bound on their behavior. Our results give bounds for the performance of our algorithm for any CFG. Additionally, for any k-ultralinear language, our results also give explicit $O(k\sqrt{n})$ and $O(2^k n^{1/3})$ additive approximations from this family which run in quadratic time. Note that as shown in our construction in the proof of hardness of parsing ultralinear grammars (Section 5.3), for any k we can restrict any context free grammar G to a k-ultralinear grammar G' such that $\mathcal{L}(G') \subseteq \mathcal{L}(G)$ contains all words that can be derived using fewer than $\leq k$ nested non-linear productions (defined formally in Appendix 5.2).

▶ **Definition 9.** For any Context Free Language edit distance approximation algorithm $\mathcal{A}$, we say that $\mathcal{A}$ is in the family $\mathcal{F}$ if it follows the same procedure as in the exact algorithm with the following modifications:

1. **Subset of non-linear productions.** $\mathcal{A}$ constructs the non-linear production edges in step 4 for the vertices in some subset of the total set of clouds $\{(p, q) \mid 1 \leq p \leq q \leq n\}$.
2. **Subset of splitting points.** For every cloud that $\mathcal{A}$ computes non-linear production edge for in step 4, when computing the edge weight c it takes minimum over only a subset of all possible splitting points.

By forgetting to construct all non-linear production edges, and by taking a minimum over fewer values when we do construct non-linear production edges, the time taken by our algorithm to construct new edges can be substantially reduced. We now give two explicit examples of how steps 1 and 2 can be implemented. We later prove explicit bounds on the approximations of these examples in Theorems 4 and 5. In both examples a *sensitivity parameter*, γ, is first chosen. We use $|OPT|$ to denote the optimum language edit distance, and $|\mathcal{A}|$ to denote the edit distance computed by an approximation algorithm $\mathcal{A}$.

▶ **Example 10.** An approximation algorithm $\mathcal{A} \in \mathcal{F}$ is a γ-*uniform grid* approximation if for $i = n, (n - \gamma), (n - 2\gamma), \dots, (n - \lfloor \frac{n}{\gamma} \rfloor \gamma)$ (see Figure 3 (left))

1. $\mathcal{A}$ constructs non-linear production edges only for an evenly-spaced $1/\gamma$ fraction of the clouds in L_i, and no others, where γ is a specified sensitivity parameter.
2. Furthermore, for every non-linear edge constructed, $\mathcal{A}$ considers only an evenly-spaced $1/\gamma$ fraction of the possible break points.

Here if i or $(n - i + 1)$ (the number of substrings of length i) is not evenly divisible by γ, we evenly space the clouds/breakpoints until no more will fit.

We will later see that the running time of such a γ-uniform grid approximation is $O(|P|(n^2 + (\frac{n}{\gamma})^3))$, and in particular for any k-ultralinear grammar G it gives an additive approximation of $O(2^k\gamma)$. Thus by setting $\gamma = n^{1/3}$, we get an $O(2^k n^{1/3})$-approximation in $O(|P|n^2)$ time (Theorem 4).

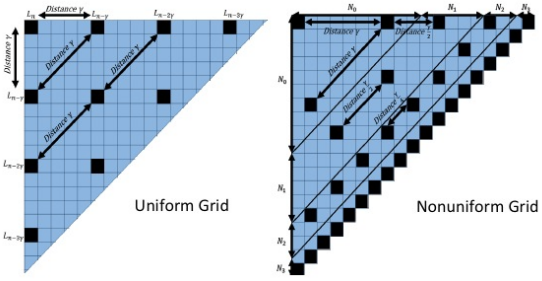

Figure 3 Non-uniform edges are computed only for a subset of the clouds (colored black). Moreover only a subset of the splitting points are considered while computing the cost.

▶ **Example 11.** For $i = 0, 1, \ldots, \log(n)$, set $N_i = \{L_j \mid \frac{n}{2^{i+1}} < j \leq \frac{n}{2^i}\}$. Let $N_i' \subset N_i$ be an evenly-spaced $\min\{\frac{2^i}{\gamma}, 1\}$-fraction of the L_j's in N_i (subset of diagonals). Then, an approximation algorithm $\mathcal{A} \in \mathcal{F}$ is a *γ-non-uniform grid* approximation if, for every $L_j \in N_i'$, $\mathcal{A}$ computes non-linear production edges only for a $\min\{\frac{2^i}{\gamma}, 1\}$-evenly-spaced fraction of the clouds in L_j. Furthermore, for any of these clouds in N_i' for which $\mathcal{A}$ does compute non-linear production edges, $\mathcal{A}$ considers only an evenly-spaced $\min\{\frac{2^i}{\gamma}, 1\}$ -fraction of all possible break points. (see Figure 3 (right))

We will see that the running time of a γ-non-uniform grid approximation is $O(|P|(n^2 + \frac{n^3}{\gamma^2}))$, and in particular for any k-ultralinear grammar, or if k is the maximum number of nested non-linear productions, it gives an additive approximation of $O(k\gamma)$. Hence setting $\gamma = \sqrt{n}$, we get an additive approximation of $O(k\sqrt{n})$ in quadratic time (Theorem 5).

4.1 Analysis.

The rest of this section will be devoted to proving bounds on the performance of approximation algorithms in $\mathcal{F}$. We use $\mathcal{T}^{OPT}$ to denote the graph which results from adding all the edges specified in the exact algorithm to $\mathcal{T}$. Recall that $\mathcal{T}$ is the graph constructed from the linear productions in G. For $\mathcal{A} \in \mathcal{F}$, we write $\mathcal{T}^{\mathcal{A}}$ to denote the graph which results from adding the edges specified by the approximation algorithm $\mathcal{A}$. Note that since $\mathcal{A}$ functions by forgetting to construct a subset of the non-linear edges created by the exact algorithm, we have that the edge sets satisfy $E(\mathcal{T}) \subseteq E(\mathcal{T}^{\mathcal{A}}) \subseteq E(\mathcal{T}^{OPT})$.

High-level Steps of the Analysis

The analysis follows the following steps.

1. We first introduce binary *production-edit trees* (PET's), a conceptual framework which will describe the execution of any algorithm in our family $\mathcal{F}$, including the exact algorithm (and more generally any language edit distance algorithm). Each node in a PET will represent a sequence of linear productions (a path in the linear edit distance graph $\mathcal{T}$), possibly including error productions. Each node will be tasked with deriving a certain substring of the input. The sequence represented by a node will terminate either when the substring has been completely derived using linear productions, or when the first non-linear production is made. Additional intuition for our definition of a PET can be found in Appendix 5.1.
2. Next we introduce functions $\alpha(p, q)$ and $\beta(p, q)$ to describe the precision of an approximation algorithm $\mathcal{A} \in \mathcal{F}$. The function $\alpha(p, q)$ gives an upper bound on the minimum ℓ_1 distance between the cloud (p, q) and the nearest cloud (r, s), $p \leq r \leq s \leq q$, such that $\mathcal{A}$

constructs non-linear edges for (r, s). The function $\beta(p, q)$ gives an upper bound on the maximum distance between the splitting points considered by $\mathcal{A}$ when constructing the non-linear edges at the cloud (r, s).

3. Finally, the goal will be to demonstrate the existence of a PET $\mathbb{T}$ *compatible* with any fixed approximation algorithm $\mathcal{A} \in \mathcal{F}$ which has a small additional cost compared to an optimal PET $\mathbb{T}_{OPT}$. To do this, we will explicitly construct $\mathbb{T}$ by transforming each node $v \in \mathbb{T}_{OPT}$ into a corresponding node $\psi(v) \in \mathbb{T}$ which *imitates* v closely. Our transformation will work inductively from the root downwards. We will bound the cost of the transformed nodes $\psi(v)$ in terms of the precision functions $\alpha(p, q)$ and $\beta(p, q)$.

Binary Production-Edit Trees.

▶ **Definition 12.** A *production-edit tree* (PET) $\mathbb{T}$ for grammar G and input string $\overline{x}$ is a binary tree which satisfies the following properties:

1. Each node of $\mathbb{T}$ stores a path in the linear graph $\mathcal{T} = \mathcal{T}(G, \overline{x})$. The path given by the root of $\mathbb{T}$ must start at the source vertex $S^{1,n}$ of $\mathcal{T}$.
2. For any node $v \in \mathbb{T}$, let $A^{p,q}, B^{r,s}$ be the starting and ending vertices of the corresponding path. If $B^{r,s}$ is not the sink t of $\mathcal{T}$, then v must have two children, v_R, v_L, such that there exists a production $B \to CD$ and the starting vertices of the paths in v_L and v_R are $C^{r,\ell}$ and $D^{\ell+1,s}$ respectively, where ℓ is some splitting point $r - 1 \leq \ell \leq s$. If $\ell = r - 1$ or $\ell = s$, then one of the children will be in the same cloud (r, s) as the ending cloud of the path given by v, and the other will be called a *nullified node*. This corresponds to the case where one of the *null* edges created in step 5 of the exact algorithm is taken.
3. If the path in $v \in \mathbb{T}$ ends at the sink of $\mathcal{T}$, then v must be a leaf in $\mathbb{T}$. If $A^{p,q}$ is the starting vertex of the path, this means that the path derives the entire substring $x_p \dots x_q$ using only linear productions. Thus a node v is a leaf of $\mathbb{T}$ if and only if it either ends at the sink or is a nullified node. It follows from 2. and 3. that every non-leaf node has exactly 2 children.

Note. Since we are now dealing with two *types* of graphs, to avoid confusion whenever we are talking about a vertex $A^{p,q}$ in any of the edit-distance graphs (such as $\mathcal{T}, \mathcal{T}^{\mathcal{A}}, \mathcal{T}^{OPT}, \mathcal{T}_{NL}$, ect), we will use the term *vertex*. When referring to the elements of a PET $\mathbb{T}$ we will use the term *node*. Also note that all error productions are linear.

Notation: To represent a node in $\mathbb{T}$ that is a path of cost c from $A^{p,q}$ to either $B^{r,s}$, or t, we will use the notation $[A^{p,q}, B^{r,s}, c]$, or $[A^{p,q}, t, c]$, respectively. If one of the arguments is either unknown or irrelevant, we write $\cdot$ as a placeholder. In the case of a nullified node, corresponding to the nullification of $A \in Q$, we write $[A, t, null(A)]$ to denote the node.

We can now represent any sequence of edits produced by a language edit distance algorithm by such a PET, where the edit distance is given by the sum of the costs stored in the nodes of the tree. To be precise, if $[\cdot, \cdot, c_1], \dots, [\cdot, \cdot, c_k]$ is the set of all nodes in $\mathbb{T}$, then the associated total cost $\|\mathbb{T}\| = \sum_{i=1}^{k} c_i$. Let $\mathcal{D}_{\mathcal{A}}$ be the set of PET's $\mathbb{T}$ compatible with a fixed approximation algorithm $\mathcal{A} \in \mathcal{F}$.

▶ **Definition 13** (PET's for $\mathcal{A}$)**.** For an approximation algorithm $\mathcal{A} \in \mathcal{F}$, let $\mathcal{D}_{\mathcal{A}}$ be the set of PET's $\mathbb{T}$ which satisfy the following constraints:

1. If $[A^{p,q}, B^{r,s}, \cdot]$ is a node in a $\mathbb{T}$, where $A, B \in Q$, then $\mathcal{A}$ must compute non-linear edges for the cloud $(r, s) \in \mathcal{T}^{\mathcal{A}}$.

2. If $[C^{r,\ell}, \cdot, \cdot], [D^{\ell+1,s}, \cdot, \cdot]$ are the left and right children of a node $[A^{p,q}, B^{r,s}, \cdot]$ respectively, then $\mathcal{A}$ must compute the splitting point ℓ for the non-linear edges in the cloud $(r, s) \in \mathcal{T}^{\mathcal{A}}$.

The set $\mathcal{D}_{\mathcal{A}}$ is then the set of all PET's which utilize only the non-linear productions and splitting points which correspond to edges that are actually constructed by the approximation algorithm $\mathcal{A}$ in $\mathcal{T}^{\mathcal{A}}$. Upon termination, any $\mathcal{A} \in \mathcal{F}$ will return the value $\|\mathbb{T}_{\mathcal{A}}\|$ where $\mathbb{T}_{\mathcal{A}} \in \mathcal{D}_{\mathcal{A}}$ is the tree corresponding to the shortest path from t to $S^{1,n}$ in $\mathcal{T}^{\mathcal{A}}$. The following theorem is not difficult to show.

▶ **Theorem 14.** *Fix any $\mathcal{A} \in \mathcal{F}$, and let c be the edit distance returned after running the approximation algorithm $\mathcal{A}$. Then if $\mathbb{T}$ is any PET in $\mathcal{D}_{\mathcal{A}}$, we have $c \leq \|\mathbb{T}\|$*

Note that since the edges of $\mathcal{T}^{\mathcal{A}}$ are a subset of the edges of $\mathcal{T}^{OPT}$ considered by an exact algorithm OPT, we also have $c \geq \|\mathbb{T}_{OPT}\|$, where $\mathbb{T}_{OPT}$ is the PET given by the exact algorithm. To prove an upper bound on c, it then suffices to construct a explicit $\mathbb{T} \in \mathcal{D}_{\mathcal{A}}$, and put a bound on the size of $\|\mathbb{T}\|$. Thus, in the remainder of our analysis our goal will be to construct such a $\mathbb{T} \in \mathcal{D}_{\mathcal{A}}$. We now introduce our precision functions.

▶ **Definition 15** (Precision Functions)**.** For any cloud $(p, q) \in \mathcal{T}^{\mathcal{A}}$, let $\alpha(p, q)$ be any upper bound on the minimum distance $d^*((p, q), (r, s)) = (r - p) + (q - s)$ such that $p \leq r \leq s \leq q$ and $\mathcal{A}$ computes non-linear edge weights for the cloud (r, s) . Let $\beta(p, q)$ be an upper bound on the maximum distance between any two splitting points which are considered by $\mathcal{A}$ in the construction of the non-linear production edges originating in a cloud (r, s) such that $\mathcal{A}$ computes non-linear edge weights for (r, s) and $d^*((p, q), (r, s)) \leq \alpha(p, q)$. Furthermore, the precision functions must satisfy $\alpha(p, q) \geq \alpha(p', q')$ and $\beta(p, q) \geq \beta(p', q')$ whenever $(q - p) \geq (q' - p')$.

While the approximation algorithms presented in this paper are deterministic, the definitions of $\alpha(p, q)$ and $\beta(p, q)$ allow the remaining theorems to be easily adapted to algorithms which *randomly select* $\mathcal{A}$ from some specified distribution over $\mathcal{F}$.

Constructing a PET $\mathbb{T} \in \mathcal{D}_{\mathcal{A}}$ similar to $\mathbb{T}_{OPT}$.

Our goal will now be to construct a PET $\mathbb{T} \in \mathcal{D}_{\mathcal{A}}$ with bounded cost. We will do this by considering each node v of $\mathbb{T}_{OPT}$ and constructing a corresponding node u in $\mathbb{T}$ such that the path stored in u *imitates* the path in v as closely as possible. A perfect imitation may not be feasible if the path at v uses a non-linear production edge in a cloud that $\mathcal{A}$ does not compute non-linear edges for. Every time this happens, we will need to move to the closest possible cloud which $\mathcal{A}$ does consider before making the *same* non-linear production that the exact algorithm did. After doing this, the ending cloud of our path will deviate from that of the optimal, so we will need to bound the total deviation that can occur throughout the construction of our tree in terms of $\alpha(p, q)$ and $\beta(p, q)$. The following lemma will be used crucially in this regard for the proof of our construction in Theorem 18. The lemma takes as input a node $[A^{p,q}, B^{r,s}, c] \in \mathbb{T}_{OPT}$ and a cloud (p', q') such that $x(p : q)$ is not disjoint from $x(p' : q')$, and constructs a path $[A^{p',q'}, B^{r',s'}, c']$ of bounded cost that is compatible with a PET $\mathbb{T} \in \mathcal{D}_{\mathcal{A}}$.

▶ **Lemma 16.** *Let $[A^{p,q}, B^{r,s}, c]$ be any non-leaf node in $\mathbb{T}_{OPT}$, and let $\mathcal{A} \in \mathcal{F}$ be an approximation algorithm with precision functions $\alpha(p, q), \beta(p, q)$. If p', q' satisfy $p \leq q'$ and $p' \leq q$, then there is a path from $A^{p',q'}$ to $B^{r',s'}$, where $r \leq r' \leq s' \leq s$, of cost $c' \leq c + (|p' - p| + |q' - q|) - (|r' - r| + |s' - s|) + 2\alpha(r, s)$ such that $\mathcal{A}$ computes non-linear production edges for cloud (r', s'). Furthermore, for any leaf node $[A^{p,q}, t, c] \in \mathbb{T}_{OPT}$, we can construct a path from $A^{p',q'}$ of cost at most $c' \leq c + (|p' - p| + |q' - q|)$ to the sink.*

We will now iteratively apply Lemma 16 to each node $v \in \mathbb{T}_{OPT}$ from the root down, transforming it into a new node $\psi(v) \in \mathbb{T}$. Here ψ will be a surjective function $\psi : V(\mathbb{T}_{OPT}) \to V(\mathbb{T})$. Lemma 16 will guarantee that the cost of $\psi(v)$ is not too much greater than that of v. If during the construction of $\mathbb{T}$, the substrings corresponding to v and $\psi(v)$ become disjoint, then we will transform the *entire* subtree rooted at v into a single node $\psi(v) \in \mathbb{T}$, thus the function may not be injective (see Definition 17).

Let v_c be any node in $\mathbb{T}_{OPT}$, and v its parent node if v_c is not the root. Let $(p, q), (r, s) \in \mathcal{T}$ and $(p_c, q_c), (r_c, s_c) \in \mathcal{T}$ be the starting and ending clouds of v and v_c respectively. Similarly let $(p', q'), (r', s')$ and $(p'_c, q'_c), (r'_c, s'_c)$ be the starting and ending clouds of $\psi(v)$ and $\psi(v_c)$ respectively. Furthermore, let (p_X, q_X) and (p'_X, q'_X), where $X = L$ for left child and $X = R$ for right child, be the starting clouds of the left and right children of v and $\psi(v)$ respectively. Let c_c be the cost of v_c, and let $\overline{c_c}$ be the cost of v_c plus the cost of all the descendants of v_c. Finally, let c'_c be the cost of $\psi(v_c)$. An abbreviated version of Theorem 18 (see the full version and proof in Appendix 5.1) relates the cost of v_c with $\psi(v_c)$ in terms of the starting and ending clouds by defining ψ inductively from the root of $\mathbb{T}_{OPT}$ down. The theorem uses Lemma 16 repeatedly to construct the nodes of $\mathbb{T}$.

▶ Theorem (abbreviated 18). For any approximation algorithm $\mathcal{A} \in \mathcal{F}$ with precision functions α, β, there exists a PET $\mathbb{T} \in \mathcal{D}_{\mathcal{A}}$ and a PET mapping $\psi : V(\mathbb{T}_{OPT}) \to V(\mathbb{T})$ such that $\mathbb{T}_{OPT}$ can be partitioned into disjoint sets $U^1_{NL} \cup U^2_{NL} \cup U^1_L \cup U^2_L \cup X$ with the following properties. For $v_c \in \mathbb{T}_{OPT}$, if $v_c \in U^1_{NL} \cup U^2_{NL}$ then v_c satisfies (Non-leaf), and if $v_c \in U^1_L \cup U^2_L$ then v_c satisfies (Leaf):

$$c'_c \le c_c + \big(|p'_c - p_c| + |q'_c - q_c|\big) - \big(|r'_c - r_c| + |s'_c - s_c|\big) + 2\alpha(r_c, s_c) \qquad \text{(Non-leaf)}$$

$$c'_c \le \overline{c_c} + |p'_c - p_c| + |q'_c - q_c| + \beta(r, s) \qquad \text{(Leaf)}$$

$$\text{Furthermore } (|p'_L - p_L| + |q'_L - q_L|) + (|p'_R - p_R| + |q'_R - q_R|) \le |r' - r| + |s' - s| + 2\beta(r, s) \quad (*)$$

Let $\mathbb{T}'_{OPT} \subset \mathbb{T}_{OPT}$ be the subgraph of nodes v in the tree for which either v is the only node mapped to $\psi(v) \in \mathbb{T}$, or v is the node closest to the root that is mapped to $\psi(v)$. In the previous theorem, the set X corresponds to the nodes v for which $\psi(v) = \psi(u)$ such that u is an ancestor of v in $\mathbb{T}_{OPT}$. So $\mathbb{T}'_{OPT} = \mathbb{T}_{OPT} \setminus X$. The final theorem is the result of summing over the bounds from Theorem 18 for all $v_j \in \mathbb{T}'_{OPT}$, applying the appropriate bound depending on the set v_j belongs to.

▶ Theorem (20). For any $\mathcal{A} \in \mathcal{F}$ with precision functions α, β, let $\mathbb{T}_{OPT}$ be the PET of any optimal algorithm. Label the nodes of $\mathbb{T}'_{OPT} \subset \mathbb{T}_{OPT}$ by $v_1 \dots v_K$. For $1 \le i \le K$, let $(p_i, q_i), (r_i, s_i)$ be the starting and ending clouds of the path v_i in $\mathcal{T}$, then

$$|OPT| \le |\mathcal{A}| \le |OPT| + \sum_{v_j \in \mathbb{T}'_{OPT}} \Big(2\alpha(r_j, s_j) + 3\beta(r_j, s_j)\Big)$$

As an illustration of Theorem 20, consider the γ-uniform grid approximation of Theorem 4. In this case, we have the upper bound $\alpha(r_j, s_j) = \beta(r_j, s_j) = 2\gamma$ for all $v_j \in \mathbb{T}_{OPT}$. Since there are k^* total vertices in $\mathbb{T}_{OPT}$, we get $|OPT| \le |\mathcal{A}| \le |OPT| + 10\gamma k^*$. To analyze the running time, note that we only compute non-linear production edges for $(n/\gamma)^2$ clouds, and in each cloud that we compute non-liner edges for we consider at most n/γ break-points. Thus the total runtime is $O(|P|(\frac{n}{\gamma})^3)$ to compute non-linear edges, and $O(|P|n^2)$ to a shortest path algorithm on $\mathcal{T}^{\mathcal{A}}$, for a total runtime of $O(|P|(n^2 + (\frac{n}{\gamma})^3))$.

Our second illustration of Theorem 20 is the γ-non-uniform grid approximation of Theorem 5. Here we obtain a $O(k\gamma)$ additive approximation in time $O(|P|(n^2 + \frac{n^3}{\gamma^2}))$. A detailed analysis can be found in Appendix 5.2.

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5 Appendix

5.1 CFG Edit Distance Approximation

Presented below are the proofs of the Lemmas and Theorems required to prove the bounds on our approximation algorithms stated in Theorems 4 and 5. We first provide additional intuition for our definition of PET's

Intuition for PET's

The intuition is as follows. The normal form we have imposed on our grammar G partitions the set of productions into **1**. linear productions, and **2.** non-linear productions with exactly two nonterminals on the right hand side. Note that error productions are linear, and therefore fall into case **1.**. Thus a derivation gives rise to a natural binary tree structure where each node represents a sequence of linear productions, and the children of a node are spawned when a non-linear production $A \to BC$ at the end of this sequence is made. Such a sequence of linear productions is given uniquely by a path of linear edges created in the exact algorithm, or equivalently by a path in the linear edit distance graph $\mathcal{T}$, which motivates the first part of our definition of a PET.

Each node will then be tasked with deriving a substring $x_p \dots x_q$ of $\overline{x}$ starting from a given non-terminal A (the root of the PET is tasked with deriving the entire string $\overline{x}$ starting from the starting symbol S). A sequence of linear productions can be made, deriving $x_p \dots x_{p'-1}$ and $x_{q'+1} \dots x_q$ and arriving at a non-terminal A', until the first non-linear production $A' \to BC$ occurs. Then the left and right child nodes are created, which will be tasked with deriving $x_{p'} \dots x_\ell$ from B and $x_{\ell+1} \dots x_{q'}$ from C respectively. This motivates the second part of our definition of PET's, which guarantees that the children of a node begin in the correct place (starting cloud). The final part of our definition ensures that the entire string $\overline{x}$ is indeed derived in the tree, whether by legal or error productions. It specifies that every node either derives the whole substring tasked to it, or makes a non-linear production such that the children are tasked with deriving what remains of the substring.

▶ Lemma (16). Let $[A^{p,q}, B^{r,s}, c]$ be any non-leaf node in $\mathbb{T}_{OPT}$, and let $\mathcal{A} \in \mathcal{F}$ be an approximation algorithm with precision functions $\alpha(p,q), \beta(p,q)$. If p', q' satisfy $p \le q'$ and $p' \le q$, then there is a path from $A^{p',q'}$ to $B^{r',s'}$, where $r \le r' \le s' \le s$, of cost $c' \le c + (|p'-p| + |q'-q|) - (|r'-r| + |s'-s|) + 2\alpha(r,s)$ such that $\mathcal{A}$ computes non-linear production edges for cloud (r', s'). Furthermore, for any leaf node $[A^{p,q}, t, c] \in \mathbb{T}_{OPT}$, we can construct a path from $A^{p',q'}$ of cost at most $c' \le c + (|p'-p| + |q'-q|)$ to the sink.

Proof. Let $e_1 \dots e_\ell$ be the sequences of edges taken by the path corresponding to $[A^{p,q}, B^{r,s}, c] \in \mathbb{T}_{OPT}$. We construct a corresponding sequence of edges $e'_1 \dots e'_\ell$ from $A^{p',q'}$, where e'_j will correspond to the same production as e_j, but with potentially higher cost. We need to consider several cases based on the overlap between $x(p:q)$ and $x(p':q')$

(1) If $p \le p'$ and $q \ge q'$, then $x(p':q')$ is a substring of $x(p:q)$. For all $1 \le j \le \ell$, if e_j produces a terminal x_ν with $p \le \nu < p'$ or $q' < \nu \le q$, set e'_j to be the deletion edge corresponding to the same production as e_j; we call these *extra deletion edges*. Otherwise set e'_j to be the same type of edge (insertion, deletion, replacement, or legal) as e_j and corresponding to the same production as e_j. Note that e'_j and e_j may be in different clouds.

Then at any point $1 \leq j \leq \ell$, after taking the edges $e'_1, \ldots, e'_j$ from $A^{p',q'}$, we will be at a vertex labeled with the same nonterminal as after taking $e_1, \ldots, e_j$ starting from $A^{p,q}$. So after taking the edges $e'_1, \ldots, e'_l$ we will arrive at a vertex B^{r^*,s^*}, where $r \leq r^* \leq s^* \leq s$.

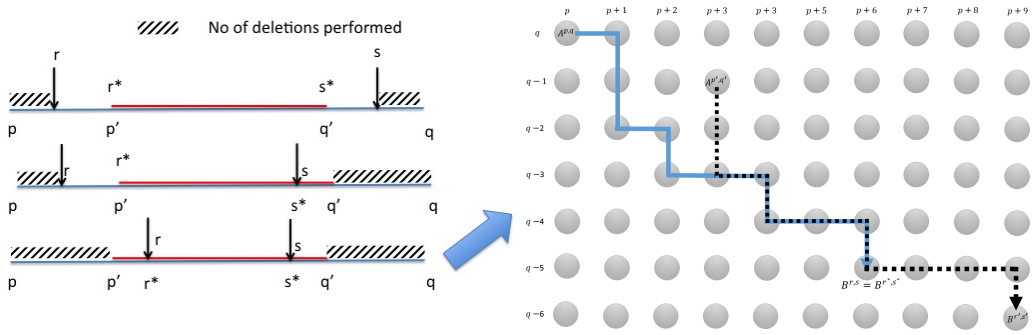


Figure 4 **(1)** $p \leq p'$ **and** $q \geq q'$: showing the number of extra deletions performed. Illustrated is an example of the path $[A^{p,q}, B^{r,s}, c]$ and the corresponding constructed path $[A^{p',q'}, B^{r',s'}, c']$ when $x(r,s)$ is a substring of $x(p',q')$. $[A^{p,q}, B^{r,s}, c]$ is given by the filled in path, and $[A^{p',q'}, B^{r',s'}, c']$ is the dotted path. Each circle is a cloud, and by construction whenever the paths meet at a cloud they will necessarily be at the same vertex within the cloud.

Now the number of extra deletion edges we add is the number of symbols x_ν with $\nu \in [p, p') \cup (q', q]$ such that x_ν is produced by an edge in $e_1 \ldots e_\ell$. Furthermore, every time we change an edge e_j that was not a deletion edge to a deletion edge e'_j, the path of OPT becomes one cloud closer to our path. This is because e_j moves OPT one cloud further, while the deletion edge e'_j remains in the same cloud. This means that for every extra deletion edge, the distance between the ending cloud of our path and the ending cloud of OPT becomes 1 less than the distance between the starting cloud of our path and the starting cloud of OPT (see Figure **??**). So the number of extra deletion edges is precisely the distance between the starting clouds minus the distance between the ending clouds, which is $(|p' - p| + |q' - q|) - (|r^* - r| + |s^* - s|)$. Thus the total cost we pay to reach B^{r^*,s^*} is at most $c + (|p' - p| + |q' - q|) - (|r^* - r| + |s^* - s|)$.

We have $(s^* - r^*) \leq (s - r)$, hence there exists a cloud (r', s') for which non-linear production edges have been computed by $\mathcal{A}$ such that $d((r^*, s^*), (r', s')) \leq \alpha(r, s)$. Thus from B^{r^*,s^*} we can take insertion edges $B^{r^*,s^*} \to \ldots \to B^{r',s'}$ arriving at the desired vertex at an additional cost of at most $\alpha(r, s)$. So the total cost is at most $c + (|p' - p| + |q' - q|) - (|r^* - r| + |s^* - s|) + \alpha(r, s)$. Since $d((r^*, s^*), (r', s')) = |r^* - r'| + |s^* - s'| \leq \alpha(r, s)$, by the triangle inequality we have $(|r^* - r| + |s^* - s|) \geq (|r' - r| + |s' - s| - \alpha(r, s))$.

Note that if the node in question is a leaf $[A^{p,q}, t, c]$, then we already have $B^{r,s} = B^{r^*,s^*} = t$ is the sink of $\mathcal{T}$, thus $(|r^* - r| + |s^* - s|) = 0$ and we pay at most $c + (|p' - p| + |q' - q|)$ to reach the sink.

Thus the total cost is at most $c + (|p' - p| + |q' - q|) - (|r' - r| + |s' - s|) + 2\alpha(r, s)$ for non-leaf nodes and $c + (|p' - p| + |q' - q|)$ for leaf nodes, the desired result.

(2) If $p > p'$ **and** $q \geq q'$. Then in this case we need to first follow some insertion edges before we can apply the argument from case **(1)**. We set $l = p - p'$ and create an edge e''_j to be the insertion edge that inserts $x_{p'+j-1}$ for $1 \leq j \leq l$. Following the edges, $e''_1, \ldots e''_l$ we pay a cost of at most $p - p'$ and arrive at a vertex $A^{p,q'}$. Note that for every insertion edge we travel across, the cloud our path is in becomes one cloud closer to the starting cloud of OPT. Now starting from $A^{p,q'}$, we are back in case **(1)** where now the distance between the beginning clouds (p, q') and (p, q) is $|q - q'|$. Thus by the argument from the first case we can

reach a vertex $B^{r',s'}$ from $A^{p,q'}$ with cost at most $c+|q'-q|-(|r'-r|+|s'-s|)+2\alpha(r,s)$ for non-leaf nodes, and cost at most $c+|q'-q|$ for leaf nodes. Since we paid at most $p-p'$ to get to $A^{p,q'}$ from $A^{p',q'}$, the total cost is at most $c+(|p'-p|+|q'-q|)-(|r'-r|+|s'-s|)+2\alpha(r,s)$ for non-leaf nodes, and $c+(|p'-p|+|q'-q|)$ for leaf nodes as desired.

(3) If $p \leq p'$ and $q < q'$. This case is symmetric to (2), as we simply start by taking edges that insert $x_{q+1},\ldots,x_{q'}$ instead of $x_{p'},\ldots,x_{p-1}$. Finally, if both $p > p'$ and $q < q'$, we take edges inserting both $x_{q+1},\ldots,x_{q'}$ and $x_{p'},\ldots,x_{p-1}$, paying a cost of $(|p'-p|+|q'-q|)$ along the way, and then we can return to case **(1)** starting at $A^{p,q}$, from which we pay a further cost of at most $c-(|r'-r|+|s'-s|)+2\alpha(r,s)$ to reach $B^{r',s'}$ for non-leaf nodes, and cost at most c to reach the sink for leaf nodes. Thus in all cases the cost is at most $c+(|p'-p|+|q'-q|)-(|r'-r|+|s'-s|)+2\alpha(r,s)$ and $c+(|r'-r|+|s'-s|)$ for non-leaf and leaf nodes respectively. Note that in all cases, every time OPT took an edge that derived a terminal in $x(p':q')$, our path also took an edge which derived the same terminal. Thus the ending clouds produced in all cases satisfy $r \leq r' \leq s' \leq s$.

◀

Now recall that our goal is to construct a PET $\mathbb{T} \in \mathcal{D}_{\mathcal{A}}$ that is compatible with the approximation $\mathcal{A}$, that has cost bounded in terms of $\mathbb{T}_{OPT}$. To do this, we will proceed by iteratively applying Lemma 16 to each node $v \in \mathbb{T}_{OPT}$ from the root down, transforming it into a new node $\psi(v) \in \mathbb{T}$. Here ψ will be a surjective function $\psi : V(\mathbb{T}_{OPT}) \to V(\mathbb{T})$. Lemma 16 will guarantee that the cost of $\psi(v)$ is not too much greater than that of v. However, during the construction of $\mathbb{T}$, the substrings corresponding to v and $\psi(v)$ may become disjoint. In this special case we will transform the *entire* subtree rooted at v into a single node $\psi(v) \in \mathbb{T}$, thus the function may not be injective. So to ease the following analysis, it will be tremendously helpful to precisely define the properties that our function must have. The following definition is therefore introduced solely for this purpose.

▶ **Definition 17.** Let $\mathbb{T}_1, \mathbb{T}_2$ be any two PET's, and $V(\mathbb{T}_1), V(\mathbb{T}_2)$ be the corresponding node sets. Then a function $\psi : V(\mathbb{T}_1) \to V(\mathbb{T}_2)$ is a *Production-Edit Tree Mapping* (PET Mapping) if it is surjective and if

1. ψ maps the root of $\mathbb{T}_1$ to the root of $\mathbb{T}_2$.
2. Every non-leaf node of $\mathbb{T}_2$ is mapped to by at most one node in $\mathbb{T}_1$.
3. If $v_1 \to v_2$ is an edge in $\mathbb{T}_1$, then either $\psi(v_1) \to \psi(v_2)$ is an edge in $\mathbb{T}_2$, or $\psi(v_1) = \psi(v_2)$.
4. If $\psi(v_1) = \psi(v_2)$ for any $v_1, v_2 \in \mathbb{T}_1$, then either v_2 is a descendant of v_1, or vice versa. Furthermore, if u is a descendant of either v_1 or v_2, then $\psi(v_1) = \psi(u) = \psi(v_2)$.

Note then that if $v_1 \neq v_2$ then $\psi(v_1) = \psi(v_2)$ can only occur if $\psi(v_1) = \psi(v_2)$ is a leaf node in $\mathbb{T}$. For such a transformation ψ, we would like the starting and ending clouds of the path given by any node $v \in \mathbb{T}_{OPT}$ to be as close as possible to those of $\psi(v)$. If we can place bounds on this distance for all pairs $v, \psi(v)$, then we show how to place bounds on the total cost of all vertices in $\mathbb{T}$. The following theorem, Theorem 18, does exactly this. For the purpose of the theorem, we will need to introduce some important notation.

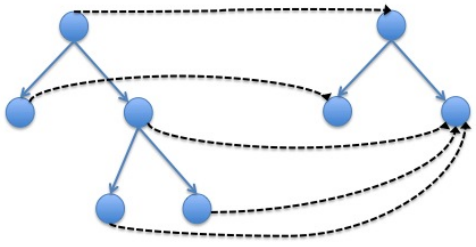


Figure 5 A PET Mapping: (left) $\mathbb{T}_{OPT} \longrightarrow \mathbb{T}$ (right)

- *Notation for mapping & nodes:* ψ will be a PET mapping (of our construction) from $V(\mathbb{T}_{OPT})$ to $\mathbb{T}$. Let v_c be any node in $\mathbb{T}_{OPT}$. If v_c is not the root then let v be the parent node of v_c.
- *Notation for starting & ending clouds:* Let $(p,q),(r,s) \in \mathcal{T}$ be the starting and ending clouds of v, and let $(p_c,q_c),(r_c,s_c) \in \mathcal{T}$ be the starting and ending clouds of v_c. Similarly, let $(p',q'),(r',s')$ and $(p'_c,q'_c),(r'_c,s'_c)$ be the starting and ending clouds of $\psi(v)$ and $\psi(v_c)$ respectively. Furthermore, let $(p_L,q_L),(p_R,q_R)$ be the starting clouds of the left and right children of v, respectively, and $(p'_L,q'_L),(p'_R,q'_R)$ the starting clouds of the left and right children of $\psi(v)$ respectively. Similarly, we denote their corresponding ending clouds by $(r_L,s_L),(r_R,s_R)$ and $(r'_L,s'_L),(r'_R,s'_R)$.
- *Notation for costs:* Let c_c be the cost of the node v_c, and let c'_c be the cost of $\psi(v_c) \in \mathbb{T}$. Write c_L, c_R for the costs of the left and right children of v, and similarly write c'_L, c'_R for the costs of the children of $\psi(v)$. Finally, for any cost c corresponding to a node u, let $\overline{c}$ denote the cost of all nodes in the subtree rooted at v.

Given a node v, the proof of the Theorem will have several cases when constructing the children of $\psi(v)$. In each case, different bounds will hold, each of which will be used in the proof of Theorem 19.

▶ **Theorem 18.** *For any approximation algorithm $\mathcal{A} \in \mathcal{F}$ with precision functions α, β, there exists a PET $\mathbb{T} \in \mathcal{D}_{\mathcal{A}}$ and a PET mapping $\psi : V(\mathbb{T}_{OPT}) \to V(\mathbb{T})$ such that $\mathbb{T}_{OPT}$ can be partitioned into disjoint sets $U^1_{NL} \cup U^2_{NL} \cup U^1_L \cup U^2_L \cup X$ with the following properties. For $v_c \in \mathbb{T}_{OPT}$, if $v_c \in U^1_{NL} \cup U^2_{NL}$ then v_c satisfies (Non-leaf), and if $v_c \in U^1_L \cup U^2_L$ then v_c satisfies (Leaf):*

$$c'_c \le c_c + \big(|p'_c - p_c| + |q'_c - q_c|\big) - \big(|r'_c - r_c| + |s'_c - s_c|\big) + 2\alpha(r_c, s_c) \qquad \text{(Non-leaf)}$$

$$c'_c \le \overline{c_c} + |p'_c - p_c| + |q'_c - q_c| + \beta(r,s) \qquad \text{(Leaf)}$$

Furthermore, if v is the parent of v_c with $v \in U^1_{NL}$, then we have

$$(|p'_L - p_L| + |q'_L - q_L|) + (|p'_R - p_R| + |q'_R - q_R|) \le |r' - r| + |s' - s| + 2\beta(r,s) \qquad (*)$$

Otherwise we have $v \in U^2_{NL}$, and if both children of v are in the set U^2_L then one of the following two inequalities is an upper bound for $c'_L + c'_R$:

$$\le \overline{c_L} + c_R + |r' - r| + |s' - s| + 2\beta(r,s) \qquad (**)$$

$$\le c_L + \overline{c_R} + |r' - r| + |s' - s| + 2\beta(r,s) \qquad (**)$$

Otherwise one of the next two inequalities is an upper bound for $c'_L + c'_R$:

$$\le \overline{c_L} + c_R + |r' - r| + |s' - s| + 2\beta(r,s) - \big(|r'_R - r_R| + |s'_R - s_R|\big) + 2\alpha(r_R, s_R) \quad (***)$$

$$\le c_L + \overline{c_R} + |r' - r| + |s' - s| + 2\beta(r,s) - \big(|r'_L - r_L| + |s'_L - s_L|\big) + 2\alpha(r_L, s_L) \quad (***)$$

Proof. We define ψ by explicitly constructing $\mathbb{T}$ from the root down. If ρ is the root of $\mathbb{T}_{OPT}$, then we first construct the root $\psi(\rho)$ of $\mathbb{T}$ via Lemma 16. We proceed inductively: supposing that $v \in \mathbb{T}_{OPT}$ is any non-leaf node with children v_L, v_R for which we have constructed $\psi(v)$ with the desired properties, we then show how to construct the children $\psi(v_L), \psi(v_R)$ of $\psi(v)$ in $\mathbb{T}$. Depending on how we construct $\psi(v_L)$ and $\psi(v_R)$ we will place v in appropriate partition (which we will specify). Formally, we will induct on the depth of $\mathbb{T}_{OPT}$. Our mapping will be ensure that the resulting tree $\mathbb{T}$ is a valid PET in $\mathcal{D}_{\mathcal{A}}$. Every non-leaf node of $\mathbb{T}$ will be explicitly constructed via Lemma 16, thus it should be noted that Lemma 16 guarantees that each time we use it to construct a path $[\cdot, B^{r',s'}, c']$ from a path $[\cdot, B^{r,s}, c]$, we will always have $r \le r' \le s' \le s$.

Building the root:

Suppose the root of $\mathbb{T}_{OPT}$ is $[S^{1,n}, B^{r_1,s_1}, c_1]$ for some $B \in Q$. We begin by mapping the root $[S^{1,n}, B^{r_1,s_1}, c_1]$ of $\mathbb{T}_{OPT}$ to the root of $\mathbb{T}$. This is a special case, since the root is neither a left nor a right child, so we only need to satisfy the (Non-leaf) property (if the root is a leaf, only linear productions are made and our approximation will be optimal). By Lemma 16, we can construct a path $S^{1,n}$ to $B^{r_1',s_1'}$, of weight at most $c_1 + 2\alpha(r_1, s_1)$ such that $\mathcal{A}$ constructs nonlinear edges in (r_1', s_1'). We thus create the root node $[S^{1,n}, B^{r_1',s_1'}, c_1']$ in $\mathbb{T}$ corresponding to this path with $c_1' \leq c_1 + 2\alpha(r_1, s_1)$, and map only the root of $\mathbb{T}_{OPT}$ to it. Then the root satisfies (Non-leaf). Now from $B^{r_1',s_1'}$ we can take the edge corresponding to the same non-linear production as the edge taken by OPT from B^{r_1,s_1}, allowing us to map a child of $[S^{1,n}, B^{r_1,s_1}, c_1]$ to a child of $[S^{1,n}, B^{r_1',s_1'}, c_1']$ such that the paths corresponding to both children begin at the same nonterminal.

Building the rest of the tree: Induction

Now suppose that we have defined ψ on all nodes in $\mathbb{T}_{OPT}$ with depth at most $i-1$ in $\mathbb{T}_{OPT}$ by constructing the nodes they map to in $\mathbb{T}$ which satisfy the desired properties, and such that the subgraph of $\mathbb{T}$ constructed so far does not yet violate any of the conditions of a valid PET. For $N, M \in Q$, let $v = [M^{p,q}, N^{r,s}, c] \in \mathbb{T}_{OPT}$ be any node at depth $i-1$, and let $\psi\big([M^{p,q}, N^{r,s}, c]\big) = [M^{p',q'}, N^{r',s'}, c'] \in \mathbb{T}$ be the node that we have mapped it to. We show how to construct both right and left children of $[M^{p',q'}, N^{r',s'}, c']$ that we can map the right and left children of $[M^{p,q}, N^{r,s}, c] \in \mathbb{T}_{OPT}$ to.

Let ℓ be the splitting point of v. In other words, the left child of $[M^{p,q}, N^{r,s}, c]$ will begin in cloud (r, ℓ) and the right child in $(\ell+1, s)$. We need to consider several cases based on the location of the splitting point ℓ and whether or not the children of $[M^{p,q}, N^{r,s}, c]$ are leaves.

Subcase 1. $r' \leq \ell < s'$**:** If v is a non-leaf and we construct the children of v under this sub case then we put v in the set U^1_{NL}. If v_c is a child of v, then if v_c is a leaf we place v_c in U^1_L.

Subcase 2. $r' > \ell$ **or** $s' < \ell$**:** If v is a non-leaf and we construct the children of v under this sub case then we put v in the set U^2_{NL}. If v_c is a child of v, then if v_c is a leaf we place v_c in U^2_L. If v_c is a *nullified node* constructed under this case, then we put all descendants of v_c in $\mathbb{T}$ in the set X.

If v is a leaf then we will have already placed it in a set when considering its parent. This covers all cases, so our inductive argument allows us to construct the entire tree $\mathbb{T}$ from the root down with the desired properties until it is a complete PET, completing the theorem. The proof of both subcases is given below. ◀

First recall our earlier notation, let $(p_L, q_L), (r_L, s_L)$ be the starting and ending clouds respectively of the left child of $v = [M^{p,q}, N^{r,s}, c]$, and similarly define $(p_R, q_R), (r_R, s_R)$ for the right child of v. Define $(p_L', q_L'), (r_L', s_L'), (p_R', q_R'), (r_R', s_R')$ similarly for the left and right children of $[M^{p',q'}, N^{r',s'}, c']$.

Subcase 1 $r' \leq \ell < s'$: The substring produced by both the left and right children of v and $\psi(v)$ overlap. Conditions of Lemma 16 hold.

Constructing the Left Child (non-leaf).

We first construct the left child of v. So suppose $[A^{r,\ell}, B^{r_L,s_L}, c_L]$ is the left child of $[M^{p,q}, N^{r,s}, c]$, coming from a production $N \to AC$, where $A, B \in Q$, and ℓ is some splitting point satisfying $r \le \ell \le s$. First suppose that $[A^{r,\ell}, B^{r_L,s_L}, c_L]$ is not a leaf. In this case we can apply Lemma 16 on $[A^{r,\ell}, B^{r_L,s_L}, c_L]$ to construct a non-leaf node v'_L. By the induction hypothesis of Theorem 18 we have $r \le r' \le s' \le s$, so there exists a splitting point ℓ' that is computed by $\mathcal{A}$ such that $|\ell - \ell'| \le \beta(r,s)$. Then we have the identity (L):

$$|r' - r_| + |\ell - \ell'| \le |r' - r| + \beta(r,s) \tag{L}$$

We now apply Lemma 16 on input node $[A^{r,\ell}, B^{r_L,s_L}, c_L]$ and starting cloud (r', ℓ') to obtain a path $v'_L = [A^{r',\ell'}, B^{r'_L,s'_L}, c'_L]$, such that

$$c'_L \le c_L + (|r' - r| + |\ell' - \ell|) - (|r'_L - r_L| + |s'_L - s_L|) + 2\alpha(r_L, s_L)$$

Notice that this is precisely the (Non-leaf) property since by definition (p'_L, q'_L) is the starting cloud of our v'_L, which is (r', l'), and (p_L, q_L) is the starting cloud of $[A^{r,\ell}, B^{r_L,s_L}, c_L]$, which is $(r, \ell) = (p_L, q_L)$. So we set $\psi([A^{r,\ell}, B^{r_L,s_L}, c_L]) = [A^{r',\ell'}, B^{r'_L,s'_L}, c'_L]$, which completes the construction of the left child.

Constructing the Right Child (non-leaf).

We now construct the right child of the same node $[N^{p',q'}, M^{r',s'}, c'] \in \mathbb{T}$ as above, using the same splitting point ℓ' as was used above. Note that the same splitting point **must** be used in both the construction of the right and left children of $[N^{p',q'}, M^{r',s'}, c']$, otherwise the resulting tree $\mathbb{T}$ would not be a valid PET. So let $[C^{\ell+1,s}, D^{r_R,s_R}, c_R]$ be the right child of $[N^{p,q}, M^{r,s}, c] \in \mathbb{T}_{OPT}$. Then applying Lemma 16 on input node $[C^{\ell+1,s}, D^{r_R,s_R}, c_R]$ and starting cloud $(\ell'+1, s')$ yields a new node $v'_R = [C^{\ell'+1,s'}, D^{r'_R,s'_R}, c'_R]$ which satisfies the (Non-leaf) property for the right child. We set $\psi([C^{\ell+1,s_{i-1}}, D^{r_i,s_i}, c_i]) = [C^{\ell'+1,s'_{i-1}}, D^{r'_i,s'_i}, c'_i]$, which completes the construction of the right child. Now since again we have $|(\ell'+1)-(\ell+1)| \le \beta(r,s)$, this gives our second identity (R):

$$|(\ell' + 1) - (\ell + 1)| + |s' - s| \le |s' - s| + \beta(r,s) \tag{R}$$

Now (L) is a bound on the distance between the starting clouds of the left child of $\mathbb{T}_{OPT}$ and the node it is mapped to by ψ, and (R) is the corresponding bound on the distance between the starting clouds of the right child and the node that it is mapped to by ψ. Adding the bounds $(L) + (R)$ produces the desired property $(*)$. Note that because equations (L) and (R) depend only on the starting cloud of the children, $(L) + (R)$ holds regardless of whether or not either child is a leaf node, since that is dictated by the ending cloud of a node. Thus we have satisfied the desired property $(*)$.

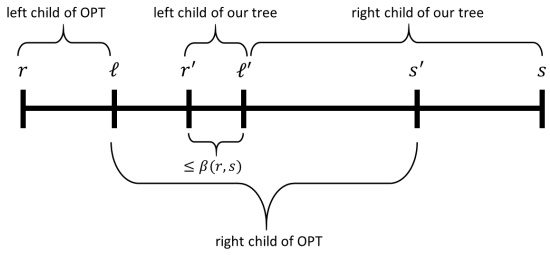
Constructing a Leaf Child. Finally, we consider the leaf case. If the left child of $[N^{p,q}, M^{r,s}, c]$ is a leaf $[A^{r,\ell}, t, c_L]$, we place $[A^{r,\ell}, t, c_L]$ in U_L^1 and then we similarly use Lemma 16 to construct the leaf $v'_L = [A^{r',\ell'}, t, c'_L]$ to map it to, such that the bound $c'_L \le c_L + (|r' - r| + |\ell' - \ell|)$ holds. Note that $\overline{c_L} = c_L$ since $[A^{r,\ell}, t, c_L]$ is a leaf, thus this bound satisfies the desired (Leaf) property. The same argument applies for when the right child is a leaf, which completes the proof of the lemma. ◀

Subcase 2 $\ell < r'$ or $\ell \ge s'$: The substrings produced by either the left or right children of v and $\psi(v)$ do not overlap. Conditions of Lemma 16 do not hold.

Recall our earlier notation: for node $v = [M^{p,q}, N^{r,s}, c] \in \mathbb{T}_{OPT}$ with children v_L and v_R, the quantity $\overline{c_L}$ is the sum of the costs of all nodes in the subtree rooted at v_L, and $\overline{c_R}$ is the

corresponding quantity for the right child v_R. In terms of our edit distance, this means that the exact algorithm derives the substring $x(p_L : q_L)$ with cost exactly $\overline{c_L}$, and derives the substring $x(p_R : q_R)$ with cost exactly $\overline{c_R}$ (and $x(p_L : q_R) = x(r : s)$).

Proof. First consider the case where $\ell < r'$. The case $\ell \geq s'$ will be an entirely symmetric argument. The difficulty in this case is that the substring (r, ℓ) which OPT derives in its left child is disjoint from the entire substring (r', s') which must be derived by the children of $[N^{p',q'}, M^{r',s'}, c'] \in \mathbb{T}$, thus we cannot utilize Lemma 16 to construct the left child. Let $[A^{r,\ell}, B^{r_L,s_L}, c_L]$ be the left child of $[M^{p,q}, N^{r,s}, c]$.



First, we fix ℓ' such that $|\ell' - r'| \leq \beta(r, s)$ and such that the splitting point ℓ' is considered by $\mathcal{A}$, which we can always find by the definition of the precision functions. We will set the left child v'_L to be $[A^{r',\ell'}, \cdot, c'_L]$, which will be a leaf in $\mathbb{T}$ because it will produce all of $x(r' : \ell')$. We will refer to such a node created in this case as a *nullified node*. Our goal is then only to bound the cost c'_i of deriving the string $x(r' : \ell')$ from A and then nullifying the remaining nonterminals. Note that all nullification edges are present in the approximate graph $\mathcal{T}^{\mathcal{A}}$, thus nullifcation of a non-terminal is always a legal procedure for any approximation algorithm in $\mathcal{F}$.

Now since we can derive $x(r' : \ell')$ with at most $\beta(r, s)$ insertions, we just need to bound the cost $null(A)$. Recall that $null(A)$ is the length of the shortest word derivable from A (which we will delete entirely). We know that the right child $[A^{r,\ell}, \cdot, c_L]$ in $\mathbb{T}_{OPT}$ derives a string of length $(\ell - r)$ starting from A with cost at most $\overline{c_L}$, where $\overline{c_L}$ is the sum of the costs of all nodes in the subtree rooted at $[A^{r,\ell}, \cdot, c_L]$. In the worst case this cost comes entirely from deletion edges, and thus $null(A) \leq \overline{c_L} + (\ell - r)$. Along with the $\beta(r, s)$ insertions, we conclude that the cost of the left child satisfies $c'_L \leq \overline{c_L} + (\ell - r) + \beta(r, s) \leq \overline{c_L} + |r' - r| + \beta(r, s) \leq \overline{c_L} + |r' - r| + |\ell' - \ell| + \beta(r, s)$, which satisfies (Leaf) (since (r, ℓ') is the starting cloud in question). We set $\psi([A^{r,\ell}, \cdot, c_L]) = [A^{r',\ell'}, \cdot, c'_L]$ and map all children in the subtree rooted at $[A^{r,\ell}, \cdot, c_L]$ to $[A^{r',\ell'}, \cdot, c'_L]$. All these descendants of $[A^{r,\ell}, \cdot, c_L]$ are placed in the set X. *Note that this is the only case where we map multiple nodes to the same place.*

We now show how to construct the right child. Let $[C^{\ell+1,s}, D^{r_R,s_R}, c_R]$ be the right child of $[N^{p,q}, M^{r,s}, c]$ in $\mathbb{T}_{OPT}$. We fix the same ℓ' as before, since the splitting points of the right and left children must agree. We now have $\ell + 1 \leq \ell' + 1 \leq s' \leq s$, thus we can apply Lemma 16 to $[C^{\ell+1,s}, D^{r_R,s_R}, c_R]$ and the cloud $(\ell' + 1, s')$ to construct a node $[C^{\ell'+1,s'}, D^{r'_R,s'_R}, c'_R]$, such that satisfies (Non-leaf) (or (Leaf) if the right child of $[N^{p,q}, M^{r,s}, c]$ is a leaf). Thus this is the desired right child. Using Lemma 16 and recalling the properties of precision functions, we have two cases: if the right child is a leaf, we have the bound:

$$c'_R \leq c_R + (s - s') + (\ell' + 1 - (\ell + 1))$$

$$\leq c_R + (s - s') + (r' - \ell) + \beta(r, s)$$

Recalling from earlier that the left child had cost c'_L such that $c'_L \leq \overline{c_L} + (\ell - r) + \beta(r, s)$, summing this bound with the bound for c'_R given above, we see that the sum of the costs of both children satisfies one of the first of the desired inequalities $(**)$.

Now if the right child is not a leaf, we have:

$$c'_R \leq c_R + (s - s') + (\ell' + 1 - (\ell + 1)) - (|r'_R - r_R| + |s'_R - s_R|) + 2\alpha(r_R, s_R)$$

$$\leq c_R + (s - s') + (r' - \ell) + \beta(r, s) - (|r'_R - r_R| + |s'_R - s_R|) + 2\alpha(r_R, s_R)$$

Summing the above inequality with the same bound on c'_L as for the leaf case gives the first of the desired inequalities $(***)$.

Case $\ell \geq s'$

This case is entirely symmetric to case where $\ell < r'$. In this case the substring $(\ell + 1, s)$ which OPT derives in its right child is again disjoint from the substring (r', s') which must be derived by the children of $[M^{p',q'}, N^{r',s'}, c'] \in \mathbb{T}$. Thus following the same procedure as above, except that we instead nullify the starting non-terminal C of the right child in $\mathbb{T}$ and then apply Lemma 16 to construct the left child, yields the desired properties. Here, the children will satisfy the second inequality of either $(**)$ or $(***)$, instead of the first as in the case above where $\ell < r'$. This completes the proof of the last case, which completes the theorem. ◀

▶ Remark. In the above proof where $\ell < r'$, or $\ell \geq s'$, and we construct a node $[A^{r'_{i-1},\ell'}, \cdot, c'_X]$ where $X \in \{R, L\}$, we are bounding c'_X as the total cost of inserting the entire substring $x(r' : \ell')$ from A and then nullifying A. Now the cost of nullifying A may involve making non-linear productions and nullifying the resulting nonterminals, so in actuality $[A^{r',\ell'}, \cdot, c'_X]$ may not be a leaf of a valid PET, it may have nullified children nodes. However, the bound we place on c'_X is an upper bound on the cost of deriving $x(r' : \ell')$ from A in the approximation graph $\mathcal{T}^{\mathcal{A}}$, and thus is a bound on the total cost of all the nodes that could be in the subtree rooted at $[A^{r',\ell'}, \cdot, c'_X]$. Therefore, the bound on c'_X satisfies the (Leaf) property, and so we can consider $[A^{r',\ell'}, \cdot, c'_X]$ to be a leaf of $\mathbb{T}$

Intuitively, the set X of the partition will be a set of nodes that we need not consider. Precisely, X consists of the set of nodes v for which $\psi(v) = \psi(u)$ such that u is an ancestor of v in $\mathbb{T}_{OPT}$. In this case, the bounds for v are *taken care of* by considering the bounds from Theorem 18 for its ancestor u. So let $\mathbb{T}'_{OPT} \subset \mathbb{T}_{OPT}$ be the subgraph of nodes v in the tree for which either v is the only node mapped to $\psi(v)$, or v is the node closest to the root that is mapped to $\psi(v)$. Then we have $\mathbb{T}'_{OPT} = \mathbb{T}_{OPT} \setminus X$. For the following proof of Theorem 19, it will be helpful to notice that the tree $\mathbb{T}'_{OPT}$ is in fact isomorphic to $\mathbb{T}$ as a graph, meaning $\psi|_{\mathbb{T}'_{OPT}}$ is an isomorphism.

▶ **Theorem 19.** *For any $\mathcal{A} \in \mathcal{F}$ with precision functions α, β, let $\mathbb{T} \in \mathcal{D}_{\mathcal{A}}$ and ψ be as constructed in Theorem* 18*, and label the nodes of $\mathbb{T}'_{OPT}$ by $v_1 \dots v_K$. For $1 \leq i \leq K$, let $(p_i, q_i), (r_i, s_i)$ be the starting and ending clouds of the path v_i in $\mathcal{T}$, and let $(p'_i, q'_i), (r'_i, s'_i)$ be the starting and ending clouds of $\psi(v_i)$. Then*

$$\|\mathbb{T}\| \leq \|\mathbb{T}_{OPT}\| + \sum_{v_j \in \mathbb{T}'_{OPT}} \Big(2\alpha(r_j, s_j) + 3\beta(r_j, s_j)\Big)$$

Proof. The above bound will be the result of summing over the bounds from Theorem 18 for all $v_j \in \mathbb{T}'_{OPT}$. The bound that applies to any given v_j is decided by the set of the partition of $\mathbb{T}'_{OPT}$ that v_j lies in. A description of each set along with the specific inequality we will use for it follows below:

- U^1_{NL} is the set of non-leaf nodes $v \in \mathbb{T}'_{OPT}$ where both children of $\psi(v) \in \mathbb{T}$ are constructed in Subcase 1 of Theorem 18. Note that the node $\psi(v)$ itself may be created either in either Subcase 1 or Subcase 2. If $\psi(v)$ is constructed in Subcase 1, we use the bound (Non-leaf), otherwise we use the bound $(***)$.
- U^2_{NL} is the set of non-leaf nodes $v \in \mathbb{T}'_{OPT}$ where the children of $\psi(v)$ are constructed in Subcase 2 of Theorem 18. If $\psi(v)$ is constructed in Subcase 1, we use the bound (Non-leaf), otherwise we use the bound $(***)$.
- U^1_L is the set of leaves $v \in \mathbb{T}'_{OPT}$ such that $\psi(v)$ was constructed in Subcase 1. We use the bound $(Leaf)$ for these nodes.
- U^2_L is the set of leaves v such that $\psi(v)$ was constructed in Subcase 2. We will not consider this set directly because the bounds for these nodes will be included in the bounds given by the parents of these nodes (either $(**)$ or $(***)$), which will be in U^2_{NL} by definition.

This covers all cases, since every time a pair of children is constructed in Subcase 2 at least one of the children must be a leaf. Now the the vertices of $\mathbb{T}'_{OPT} = \mathbb{T}_{OPT} \setminus X$ are in bijection with those of $\mathbb{T}$ via ψ, so by considering the inequalities which upper bound the cost of $\psi(v)$ in terms of the cost of v, taken over all v in $\mathbb{T}'_{OPT}$, we obtain an upper bound for the total cost of our tree $\mathbb{T}$. For any non-leaf $v_j \in U^1_{NL} \cup U^2_{NL}$, let R_j, L_j be the indices of its right and left children v_{R_j}, v_{L_j}. For any non-root node $v_j \in \mathbb{T}'_{OPT}$, let P_j be the index of its parent v_{P_j}. Then summing the inequalities which apply to each $v_j \in \mathbb{T}'_{OPT}$ we obtain:

$$\|\mathbb{T}\| \leq \|\mathbb{T}_{OPT}\| + \mathcal{W}^1_{NL} + \mathcal{W}^2_{NL} + \mathcal{W}_L$$

The rest of the proof will be spent analyzing these new terms. Firstly, $\mathcal{W}^1_{NL} =$

$$\sum_{v_j \in U^1_{NL}} \Big(|p'_{R_j} - p_{R_j}| + |q'_{R_j} - q_{R_j}| + |p'_{L_j} - p_{L_j}| + |q'_{L_j} - q_{L_j}| - \big(|r'_j - r_j| + |s'_j - s_j|\big) + 2\alpha(r_j, s_j)) \Big)$$

$$\mathcal{W}^2_{NL} = \sum_{v_j \in U^2_{NL}} \Big(\big(|r'_j - r_j| + |s'_j - s_j|\big) + 2\beta(r_j, s_j) - \big(|r'_j - r_j| + |s'_j - s_j|\big) + 2\alpha(r_j, s_j)) \Big)$$

$$\mathcal{W}_L = \sum_{v_j \in U^1_L} \Big(\beta(r_{P_j}, s_{P_j}) \Big)$$

First sum:

We first examine $\mathcal{W}^1_{NL}$. For each node $v_j \in U^1_{NL}$ if v_j was created in Subcase 1 of Theorem 18 then the bound (Non-leaf) applies. If v_j was created in Subcase 2, then one of the two bounds $(***)$ will hold for the cost of v_j (depending on whether v_j is a right or left child). Note that if this is the case then the sibling node of v_j must be a leaf (since in Subcase 2 at least one of the children is always a leaf). Either way the bound $(*)$ applies to the children of v_j since they were created in Subcase 1. Now in either case we have put the portion $\big(- \big(|r'_j - r_j| + |s'_j - s_j|\big) + 2\alpha(r_j, s_j)$ of the (Non-leaf) or $(***)$ bound for v_j next to the positive portions of the bounds which come from its children (either (Non-leaf) or (Leaf)) $|p'_{R_j} - p_{R_j}| + |q'_{R_j} - q_{R_j}| + |p'_{L_j} - p_{L_j}| + |q'_{L_j} - q_{L_j}|$, in the same summand. Observe $(***)$ is a bound not just on v_j but also on its sibling, so we must make sure we do not use the portion $\big(- \big(|r'_j - r_j| + |s'_j - s_j|\big) + 2\alpha(r_j, s_j)$ of the bound twice. But this cannot happen,

since the sibling of v_j must be a leaf in U_L^2 (it cannot be in U_{NL}^1 or U_{NL}^2), so we will not sum this same portion of its bound again.

Now the goal of organizing the summands like this is to easily apply Property $(*)$ of Theorem 18 (this holds since $v_j \in U_{NL}^1$) which gives $|p'_{R_j} - p_{R_j}| + |q'_{R_j} - q_{R_j}| + |p'_{L_j} - p_{L_j}| + |q'_{L_j} - q_{L_j}| - \big(|r'_j - r_j| + |s'_j - s_j|\big) \leq 2\beta(r_j, s_j)$. *Thus the bound on the cost of the children of any node v_j is split between the summands corresponding to them and the summands corresponding to v_j.* If the bound for v_j was (Non-leaf), then the rest of the bound for v_j will be put in the sum $\mathcal{W}_{NL}^1$ for the parent of v_j. If the bound for v_j was one of the $(***)$ bounds, then the rest of the bound will be put in the sum $\mathcal{W}_{NL}^2$ for the parent of v_j. Note in the special case of the root we have $|p'_1 - p_1| + |q'_1 - q_1| = 0$, and so this does not appear in the sum $\mathcal{W}_{NL}^1$. Then using Property $(*)$, gives

$$\mathcal{W}_{NL}^1 \leq \sum_{v_j \in U_{NL}} \Big(2\alpha(r_j, s_j) + 2\beta(r_j, s_j)\Big)$$

Second sum:

We now consider $\mathcal{W}_{NL}^2$. For each node $v_j \in U_{NL}^2$ as in the first sum either the bound (Non-leaf) or $(***)$ applies to v_j, and either the bound $(**)$ or $(***)$ applies to the children of v_j since they were created in Subcase 2. Again, in either case we put the portion $\big(-\big(|r'_j - r_j| + |s'_j - s_j|\big) + 2\alpha(r_j, s_j)\big)$ of the bound that applies to v_j next to the portion $\big(|r'_j - r_j| + |s'_j - s_j|\big) + 2\beta(r_j, s_j)$ of the bounds either $(**)$ or $(***)$ that apply to the children of v_j. Canceling terms in the definition of $\mathcal{W}_{NL}^2$ above gives:

$$\mathcal{W}_{NL}^2 = \sum_{v_j \in U_{NL}^2} \Big(2\beta(r_j, s_j) + 2\alpha(r_j, s_j))\Big)$$

Third sum:

$\mathcal{W}_L$ simply accounts for the fact that for every leaf node $v \in U_L^1$, the portion of the sum $(Leaf)$ that applies to v which is included in the sum for its parent, either in $\mathcal{W}_{NL}^1$ or $\mathcal{W}_{NL}^2$, does not include the $\beta(r_{P_j}, s_{P_j})$ term. All other leaf nodes are in U_L^2 (created in Subcase 2), and the bounds for these nodes are already accounted for by the bounds for their parents in W_{NL}^2. Since the size of U_L^1 can be on the order of the number of all nodes, we give the following bound:

$$\mathcal{W}_L = \sum_{v_j \in U_L^1} \Big(\beta(r_{P_j}, s_{P_j})\Big) \leq \sum_{v_j \in \mathbb{T}'_{OPT}} \beta(r_j, s_j)$$

So all together $\mathcal{W}_{NL}^1 + \mathcal{W}_{NL}^2 + \mathcal{W}_L \leq \sum_{v_j \in \mathbb{T}'_{OPT}} \big(2\alpha(r_j, s_j) + 3\beta(r_j, s_j)\big)$, which completes the proof.

◀

▶ **Theorem 20.** *For any $\mathcal{A} \in \mathcal{F}$ with precision functions α, β, let $\mathbb{T}_{OPT}$ be the PET of any optimal algorithm. Label the nodes of $\mathbb{T}'_{OPT} \subset \mathbb{T}_{OPT}$ by $v_1 \dots v_K$. For $1 \leq i \leq K$, let $(p_i, q_i), (r_i, s_i)$ be the starting and ending clouds of the path v_i in $\mathcal{T}$, and let $(p'_i, q'_i), (r'_i, s'_i)$ be the starting and ending clouds of $\psi(v_i)$. Then*

$$|OPT| \leq |\mathcal{A}| \leq |OPT| + \sum_{v_j \in \mathbb{T}'_{OPT}} \Big(2\alpha(r_j, s_j) + 3\beta(r_j, s_j)\Big)$$

Proof. The result follows immediately from Theorems 14 and 19.

◀

5.2 Two Explicit Edit Distance Approximation Algorithms

Theorem 20 allows us to place bounds on the additive approximation in terms of the structure of the optimum derivation tree. For each non-linear production made in such a derivation, another term is accumulated in the bound given in 20, thus in general the approximation performs better for derivations which have a high ratio of linear to non-linear productions. Additionally, we will demonstrate bounds which depended on the depth of the PET, which we call the maximum number of *nested non-linear productions* of the derivation. We now formalize this notion using only the language of CFG's. Fix a derivation of a string $w \in \mathcal{L}(G)$, and fix any two specific appearances of non-terminals A, B within the derivation (note that a given non-terminal can appear multiple times in one derivation). We say that B follows from A if in the derivation of w a (possibly empty) sequence of linear productions is first made starting from A, and then a non-linear production is made which produces B.

▶ **Definition 21.** For any derivation of string $w \in \mathcal{L}(G)$, we define the maximum number of *nested non-linear productions* to be the maximum length of a sequence $A_1, A_2, \ldots, A_k$ such that A_i follows from A_{i-1} for $i = 2, 3, \ldots, k$ in the derivation of w. Equivalently this is the depth of a PET for this derivation of w.

For example, if G is the well-known *Dyck Languages* [39, 7], the language of well-balanced parenthesis, then this quantity is the number of levels of nesting in w. Note that a string consisting of n open parenthesis followed by n matching closed parenthesis has zero levels of nesting, whereas the string "((())())" has one. Intuitively, if we folded all the matching parenthesis onto each other, we are concerned with the maximum number of *nested folds*. As an another example, consider RNA-folding [10, 43, 49] which is a basic problem in computational biology and can be modeled by grammars. Here the maximum number of nested non-linear productions is the maximum number of nested folds in w.

Now we fix any input string $\overline{x} \in \Sigma^*$ and CFG G. Let $w \in \mathcal{L}(G)$ be any string with minimum edit distance to $\overline{x}$, and let k be the number of nested non-linear productions in any derivation of w, and let k^* be the total number of non-linear productions. Equivalently, k is the depth of an optimal PET for $\overline{x}$, and k^* is the total number of nodes in this PET. Note if G is k'-ultralinear in our normal form, we have the natural bounds $k \leq k'$ and $k^* \leq 2^{k'}$. We now give two specific approximation algorithms based on constructions defined in Section 4. For any $\mathcal{A}$, we write $|\mathcal{A}|$ to denote the edit distance returned by $\mathcal{A}$, and $|OPT|$ for the edit distance returned by an exact algorithm.

▶ Theorem (4). If $\mathcal{A}$ is a γ-uniform grid approximation, then $\mathcal{A}$ produces a value $|\mathcal{A}| = |\mathbb{T}_{\mathcal{A}}|$ such that

$$|OPT| \leq |\mathcal{A}| \leq |OPT| + O(k^*\gamma)$$

in $O(|P|(n^2 + (\frac{n}{\gamma})^3))$ time.

Proof. In this case, we have the upper bound $\alpha(p, q) = \beta(p, q) = 2\gamma$ for all $1 \leq i \leq k$ and $1 \leq p \leq q \leq n$. Since there are k^* vertices in the optimal PET $\mathbb{T}_{OPT}$, it follows from Theorem 20 that:

$$|OPT| \leq |\mathcal{A}| \leq |OPT| + k^*(10\gamma)$$

Runtime.

We only compute non-linear production edges for $(n/\gamma)^2$ clouds. For each non-linear production, we compute corresponding to a substring of size m, at most $m/\gamma \leq n/\gamma$ break-points. Thus the total runtime is $O(|P|(\frac{n}{\gamma})^3)$ to compute non-linear edges, and $O(|P|n^2)$ to run a single source shortest path algorithm from the sink to all vertices of $\mathcal{T}^{\mathcal{A}}$, for a total runtime of $O(|P|(n^2 + (\frac{n}{\gamma})^3))$

◀

▶ Theorem (5). Let $\mathcal{A}$ be any γ-*non-uniform grid* approximation, then $\mathcal{A}$ produces a value $|\mathcal{A}| = |\mathbb{T}_{\mathcal{A}}|$ such that

$$|OPT| \leq |\mathcal{A}| \leq |OPT| + O(k\gamma)$$

in $O\big(|P|\big(n^2 + \frac{n^3}{\gamma^2}\big)\big)$ time.

Proof. Let $V_i = \{v_1, \dots v_l\}$ be the set of all vertices at depth i in $\mathbb{T}_{OPT}$, and let w_j be the substring corresponding to the ending cloud (r_j, s_j) of v_j for $1 \leq j \leq l$. Then $w_1, w_2, \dots w_l$ are the substrings which our algorithm derives from nonterminals at depth lower than i in $\mathbb{T}_{OPT}$. If $\frac{n}{2^{t+1}} \leq |w_j| \leq \frac{n}{2^t}$, we have $(r_j, s_j) \in N_t$ and can therefore set the upper bound on the precision functions $\alpha(r_j, s_j) = \beta(r_j, s_j) = \frac{\gamma}{2^t}$. Since the substrings must be disjoint, clearly $\sum_{j=1}^{l} |w_j| \leq n$. Set $|v_j| = 2\alpha(r_j, s_j) + 3\beta(r_j, s_j)$. Then if $(r_j, s_j) \in N_t$, we have $|v_j| \leq 5\frac{\gamma}{2^t}$. Then $|v_j| \leq (2|w_j|)5\frac{\gamma}{n}$. We have

$$\sum_{v_j \in V_i} |v_j| \leq \sum_{j=1}^{l} 10|w_j|\frac{\gamma}{n} \leq 10(n)\frac{\gamma}{n} = 10\gamma = O(\gamma)$$

Note that this bound is independent of the depth i. Since $\mathbb{T}_{OPT}$ has depth k by definition, Theorem 20 states that the additive error is at most $\sum_{i=1}^{k} \sum_{v \in V_i} |v|$. As just shown, each inner sum is at most 10γ, thus the total additive error is $10k\gamma = O(k\gamma)$ as desired.

Runtime.

Let $N_i'' \subset N_i' \subset N_i$ be the set of clouds which we actually construct non-linear edges for. There are at most $\frac{n}{2^{i+1}}$ L_j's in N_i, and each has at at most n clouds in it. Then if $2^i \leq \gamma$ we have:

$$|N_i''| \leq n\frac{n}{2^{i+1}}(\frac{2^i}{\gamma})^2 = \frac{n^2 2^{i-1}}{\gamma^2}$$

Now if $2^i > \gamma$, then we consider all clouds in N_i. In other words, if $j \leq \frac{n}{\gamma}$, then we consider all clouds in L_j. Now $|L_j| \leq n$ for all j, so we consider at most $\sum_{j=1}^{n/\gamma} |L_j| \leq \frac{n^2}{\gamma}$ in this second section. Thus in $\mathcal{T}^{\mathcal{A}}$ we create nonlinear edges for a total of

$$\sum_{i=1}^{\log(\gamma)} |N_i''| + \frac{n^2}{\gamma} \leq \sum_{i=1}^{\log(\gamma)} \frac{n^2 2^{i-1}}{\gamma^2} + \frac{n^2}{\gamma} \leq 2\frac{n^2}{\gamma}$$

clouds. Now since for every substring of length $\ell \leq \frac{n}{2^i}$, we consider at most $\ell\frac{2^i}{\gamma} \leq \frac{n}{\gamma}$ breakpoints the total number of breakpoints considered over all nodes in the first section

(where $2^i \leq \gamma$) is at most $\frac{n}{\gamma}\sum_{i=1}^{\log(\gamma)} |N_i''| = \frac{n^3}{\gamma^2}$. For the second section with strings of length $\leq \frac{n}{\gamma}$, the number of breakpoints is at most

$$|L_1|(1) + |L_2|(2) + \ldots |L_{n/\gamma}|\frac{n}{\gamma} = n(1) + (n-1)2 + \cdots + (n - \frac{n}{\gamma})\frac{n}{\gamma} \leq n\sum_{i=1}^{n/\gamma} i = O(\frac{n^3}{\gamma^2})$$

Thus a total of $O(\frac{n^3}{\gamma^2})$ breakpoints are considered for each production in $|P|$ while constructing $\mathcal{T}^{\mathcal{A}}$. There are $O(|P|n^2)$ edges in $\mathcal{T}^{\mathcal{A}}$, thus running single source shortest path takes a total of $|P|n^2$ time, so the total runtime is $O\big(|P|\big(n^2 + \frac{n^3}{\gamma^2}\big)\big)$.

◀

5.3 Hardness Results

Linear Grammar Edit Distance Hardness

A recent result of Backurs and Indyk [6] has demonstrated that finding a truly subquadratic algorithm for computing the exact edit distance between two binary strings would imply the falsity of the Strong Exponential Time Hypothesis (SETH). This result has been shown to hold even in the case where the input strings are binary [11]. We extend this SETH-hardness result by demonstrating that a truly subquadratic algorithm for linear language edit distance to a grammar of constant size would imply a truly subquadratic algorithm for binary string edit distance.

We first define the linear grammar $G = (Q, \Sigma, P, S)$ with nonterminals $Q = \{S, S_z\}$, alphabet $\Sigma = \{0, 1\} \cup \{z\}$, and productions:

$$S \to 0S0 \mid 1S1 \mid \mid S_z$$

$$S_z \to zS_zz \mid z$$

Given two binary strings, $A \in \{0,1\}^n$ $B \in \{0,1\}^m$, let B^R be the reverse of B. Then define:

$$w_{A,B} = A\overline{z}_{n+m}B^R$$

Where $\overline{z}_{n+m} = \oplus_{i=1}^{m+n} z$, and $\oplus$ is concatenation. Note that our grammar has constant size. The language it recognizes is precisely $\mathcal{L}(G) = \{C\overline{z}_t C^R \mid C \in \{0,1\}^k, k, t \geq 0\}$, which is the set of all binary palindromes separated by any number of z's in between. The purpose of the dummy variable z will be to avoid *cheating* in the editing procedure by blurring the line between which string is A and which is B^R in a valid editing procedure.

▶ **Theorem 22.** *The language edit distance between $w_{A,B}$ and G is equal to the string edit distance between A and B.*

Proof. Let $d(w_{A,B}, G)$ be the language edit distance, and let $d(A, B)$ be the string edit distance between A and B. We first show $d(w_{A,B}, G) \leq d(A, B)$. Let $\tilde{e}_A$ be an *ordered* sequence of insertions, deletions, and substitutions which edit A into B with minimum cost. Then applying the editing procedure to the substring $w_{A,B}(1:n)$ transforms $w_{A,B}$ into the string $B\overline{z}_{n+m}B^R$ with equal cost. Since $B\overline{z}_{n+m}B^R \in \mathcal{L}(G)$, we have $d(w_{A,B}, G) \leq d(A, B)$.

Now let $\tilde{e}_G$ be an *ordered* sequence of edits of minimum cost which modify $w_{A,B}$ into a valid word of $\mathcal{L}(G)$. We first argue that the substring $\overline{z}_{n+m}$ is not modified. Since $\mathcal{L}(G)$ admits a string $C\overline{z}_t C^R$ for any t, deleting any of the z's in $w_{A,B}$ would not decrease the Levenshtein distance between the input string and the language, *unless* potentially in the case that all $n + m$ of the z's were deleted. But clearly $D_{A,B} \leq n + m$, thus we need not

consider this case. Similarly, inserting z's can never be helpful. Replacing one of the z's with a 0 or 1, or inserting a 0/1 within the substring $\overline{z}_{n+m}$, say at position j, would then force that all z's in position either $< j$ or $> j$ be deleted or replaced. We have established that deleting z's is never helpful, and the effect of replacing a string of z's from j to the ending of A or the start of B^R could be equivalently accomplished by inserting the same terminals between $\overline{z}_{n+m}$ and A or B^R respectively. Thus we can assume that the substring $\overline{z}_{n+m}$ is never modified by $\tilde{e}_G$.

Then $\tilde{e}_G$ can be partitioned into the set of edits made to the substring A, and the edits made to B^R. This gives a valid procedure to edit A into C and B into C for some $C \in \{0,1\}^k$ and k. Since edit distance is a metric on the space of strings, we have $d(A,B) = \min_{k, C \in \{0,1\}^k} d(A,C) + d(C,B)$. But we have just shown that the left hand side is at most $d(w_{A,B}, G)$, which completes the proof.

◀

▶ **Theorem 23** (Hardness of Linear Language Edit Distance). *There exists no algorithm to compute the minimum edit distance between a string $\overline{x}$, $|\overline{x}| = n$ and a linear language $\mathcal{L}(G)$ in $o(n^{2-\epsilon})$ time for any constant $\epsilon > 0$, unless SETH is false.*

Proof. The theorem follows immediately from 22 and from the results of [11]. ◀

Ultralinear Language Parsing Hardness

A recent result of Abboud, Backurs and Williams [3] has shown that any algorithm which can solve the recognition problem for an input string of length n to a context free grammar $\mathcal{G}$ in time $O(n^F)$ can be modified to an algorithm which can solve the $3k$-clique problem on a graph of order n in time $O(n^{Fk})$. A well known conjecture of graph algorithms states that the smallest such value of F for which $3k$-clique can be solved is 3 for combinatorial algorithms, and ω for any algorithm, where ω is the exponent of fast matrix multiplication. A refutation of this conjecture would additionally result in faster exact algorithms for Max-Cut [47, 45], among other consequences.

The proof of hardness in [3] proceeds by enumerating all k-cliques in an input graph, and then judiciously constructing an input string w over an alphabet Σ which encodes all of these k-cliques. A grammar $\mathcal{G}$ of constant size is then introduced such that $\mathcal{G}$ accepts w if and only if the input graph contains a $3k$-clique.

In this section we adapt this approach so that the grammar in question is ultralinear. We do this by constructing an ultralinear grammar $\mathcal{G}_U^\ell$, parameterized by a constant ℓ, such that $\mathcal{L}(\mathcal{G}_U^\ell) \subset \mathcal{L}(\mathcal{G})$ and such that if w is a string constructed from a graph G as specified by [3], then G has a $3k$-clique if and only if $w \in L(\mathcal{G}_U^\ell)$. Our grammar is essentially $\mathcal{G}$, but with modifications made in order to bound the total number of non-linear productions which can be made during any derivation. Our grammar will have size $O(\ell) = O(n^3)$, but since $|w| \in O(k^2 n^{k+1})$ and the blowup in grammar size is independent of k, this is not problematic. It follows that if the currently known clique algorithms are optimal, the recognition problem for ultralinear grammars cannot be solved in $o(Poly(|\mathcal{G}|)n^F)$ time, where F is as in the conjecture above. We present our adaptation $\mathcal{G}_U^\ell$ below.

▶ **Theorem 24** (Hardness of Ultralinear Grammar Parsing). *There is a ultralinear grammar $\mathcal{G}_U^\ell = \mathcal{G}_U$ such that if we can solve the membership problem for string of length n in time $O(|\mathcal{G}_U|^\alpha n^c)$, where $\alpha > 0$ is some fixed constant, then we can solve the k-clique problem on a graph with n nodes in time $O(n^{c(k+3)+3\alpha})$.*

Encoding of the Graph

Let G be a graph on n vertices. For every vertex $v \in V(G)$, let $\overline{v}$ be a unqiue binary encoding of v of size exactly $2\log(n)$. Let $N(v)$ be the neighborhood of v. We define a set of gadgets, which are exactly those introduced in [3], over the same alphabet $\Sigma = \{0, 1, \$, \#, a_{start}, a_{mid}, a_{end}, b_{start}, b_{mid}, b_{end}, c_{start}, c_{mid}, c_{end}\}$. Firstly are the so-called *node* and *list* gadgets:

$$NG(v) = \#\overline{v}\# \qquad LG(v) = \# \bigoplus_{u \in N(v)} (\$\overline{u}^R\$)\#$$

where $\overline{u}^R$ is the reverse of $\overline{u}$. We then enumerate all k-cliques in G, and use $\mathcal{C}_k$ to denote the set of all k-cliques in G. Let $t = \{v_1, \ldots, v_k\} \in \mathcal{C}_k$ be any k-clique. Then the so-called "clique-node" and "clique-list" gadgets are given by

$$CNG(t) = \bigoplus_{v \in t} (NG(v))^k$$

$$CLG(t) = \Big(\bigoplus_{v \in t} LG(v) \Big)^k$$

Along with the additional three gadgets

$$CG_\alpha(t) = a_{start} CNG(t) a_{mid} CNG(t) a_{end}$$

$$CG_\beta(t) = b_{start} CLG(t) b_{mid} CNG(t) b_{end}$$

$$CG_\gamma(t) = c_{start} CLG(t) c_{mid} CLG(t) c_{end}$$

Finally, the encoding of the G into a string w is given by

$$w = (\bigoplus_{t \in \mathcal{C}_k} CG_\alpha(t)))(\bigoplus_{t \in \mathcal{C}_k} CG_\beta(t))(\bigoplus_{t \in \mathcal{C}_k} CG_\gamma(t)))$$

Note that $|w| \in O(k^2 n^{k+1})$, and the cost of constructing the string w is linear in its length.

The Ultralinear Grammar

Our grammar $\mathcal{G}_U^\ell = (Q, \Sigma, P, S)$ is given by

$$Q = \Big(\bigcup_{i=1}^{\ell} \{\mathbf{V}^i_{\alpha\gamma}, \mathbf{V}^i_{\alpha\beta}, \mathbf{V}^i_{\beta\gamma}, \mathbf{S}^i_{\alpha\gamma}, \mathbf{S}^i_{\alpha\beta}, \ \mathbf{S}^i_{\beta\gamma}, \mathbf{N}^i_{\alpha\gamma}, \mathbf{N}^i_{\alpha\beta}, \mathbf{N}^i_{\beta\gamma}\} \Big) \bigcup \{\mathbf{S}, \mathbf{S}^*_{\alpha\gamma}, \mathbf{S}^*_{\alpha\beta}, \mathbf{S}^*_{\beta\gamma} \mathbf{W}, \mathbf{W}', \}$$

where i ranges from $i = 1, 2, \ldots, \ell$, for some ℓ which we will later fix. The main productions are:

$$\mathbf{S} \to \mathbf{W} a_{start} \mathbf{S}^1_{\alpha\gamma} c_{end} \mathbf{W} \qquad \mathbf{S}^*_{\alpha\gamma} \to a_{mid} \mathbf{S}^1_{\alpha\beta} b_{mid} \mathbf{S}^1_{\beta\gamma} c_{mid}$$

$$\mathbf{S}^*_{\alpha\beta} \to a_{end} \mathbf{W} b_{start} \qquad \mathbf{S}^*_{\beta\gamma} \to b_{end} \mathbf{W} c_{start}$$

Then for $xy \in \{\alpha\beta, \alpha\gamma, \beta\gamma\}$, and for $i = 1, 2, \ldots, \ell - 1$, we have the "$xy$-listing rules":

$$\mathbf{S}^i_{xy} \to \mathbf{S}^*_{xy} \qquad \mathbf{S}^i_{xy} \to \#\mathbf{N}^{i+1}_{xy}\$\mathbf{V}^{i+1}_{xy}\#$$
$$\mathbf{N}^i_{xy} \to \#\mathbf{S}^{i+1}_{xy}\#\mathbf{V}^{i+1}_{xy}\$ \qquad \mathbf{N}^i_{xy} \to \sigma\mathbf{N}^i_{xy}\sigma$$

Where $\sigma \in \{0, 1\}$. Finally, again for $i = 1, 2, \ldots, \ell - 1$, we have the "assisting rules":

$$\mathbf{W} \to \epsilon \mid \lambda\mathbf{W} \qquad \mathbf{W}' \to \epsilon \mid \sigma\mathbf{W}' \qquad \mathbf{V}^i_{xy} \to \epsilon \mid \$\mathbf{W}'\$\mathbf{V}^{i+1}_{xy}$$

For all $\lambda \in \Sigma$ and $\sigma \in \{0, 1\}$. Then for $i = 1, \ldots, \ell - 1$, the partition

$$Q_{2\ell} = \{\mathbf{S}\},\ Q_{2\ell-i} = \{\mathbf{S}^i_{\alpha\gamma}, \mathbf{N}^i_{\alpha\gamma}, \mathbf{V}^i_{\alpha\gamma}\},\ Q_\ell = \{\mathbf{S}^*_{\alpha\gamma}\}$$

$$Q_{\ell-i} = \{\mathbf{S}^i_{\alpha\beta}, \mathbf{S}^i_{\beta\gamma}, \mathbf{N}^i_{\alpha\beta}, \mathbf{N}^i_{\beta\gamma}, \mathbf{V}^i_{\alpha\beta}, \mathbf{V}^i_{\beta\gamma}\},\ Q_1 = \{\mathbf{S}^*_{\alpha\beta}, \mathbf{S}^*_{\beta\gamma}\}, Q_0 = \{\mathbf{W}, \mathbf{W'}\}$$

satisfies the ultra-linear property. Our grammar is the same as that in [3], except we replace the set of nonterminals $\{\mathbf{V}, \mathbf{S}_{\alpha\gamma}, \mathbf{S}_{\alpha\beta},\ \mathbf{S}_{\beta\gamma}, \mathbf{N}_{\alpha\gamma}, \mathbf{N}_{\alpha\beta}, \mathbf{N}_{\beta\gamma}\}$ by ℓ *identical* copies, each with a index in $\{1, \ldots, \ell\}$, such that every time one copy of a nonterminal in this set is produced from another via a non-linear production, the resulting copy has a strictly greater index. Note that we replace the $\mathbf{V}$ of [3] with 3 further copies $\{\mathbf{V}^i_{\alpha\gamma}, \mathbf{V}^i_{\alpha\beta}, \mathbf{V}^i_{\beta\gamma}\}$ for each $i = 1, \ldots, \ell$, such that $\mathbf{V}^i_{xy}$ can only be produced by $\mathbf{N}^{i-1}_{xy}, \mathbf{S}^{i-1}_{xy}$, and $\mathbf{V}^{i-1}_{xy}$ (the nonterminals with the same subscript), as opposed the $\mathbf{V}$ in [3] which could be produced by $\{\mathbf{N}_{xy}, \mathbf{S}_{xy}, V\}$ for any $xy \in \{\alpha\beta, \alpha\gamma, \beta\gamma\}$.

Finally, for every copy of such a nonterminal with index ℓ, we prevent this nonterminal from making any further non-linear productions. Doing this places a strict limit on the maximum number of times a given non-linear productions may be used, in order to preserve the ultralinear property. Since we have not added any new productions, but instead modified each non-linear production of [3] such that it cannot be used more than ℓ times, we have that $\mathcal{L}(\mathcal{G}^\ell_U) \subset \mathcal{L}(\mathcal{G})$. Thus, the language recognized by our grammar is strictly a subset of the language recognized by the context free grammar $\mathcal{G}$, which consists of strings which can be produced with arbitrarily many non-linear productions. Specifically, as $\ell \to \infty$, the language $\mathcal{L}(\mathcal{G}^\ell_U)$ becomes precisely $\mathcal{L}(\mathcal{G})$. Note that the number of nonterminals and the number of productions in $\mathcal{G}^\ell_U$ is linear in the size of $\mathcal{G}^\ell_U$, thus we have $|\mathcal{G}^\ell_U| = O(\ell)$

We will show that taking $\ell = O(n^3)$ will be sufficient in order to recognize any encoding w of a graph G which contains a $3k$-clique, which will prove 24. Our proof is essentially the same as that of [3], except we count the number of times that non-linear productions must be used to derive a string w which encodes a $3k$-clique.

▶ **Theorem 25.** *Let w be an encoding of a graph G as given above. Then $\mathcal{G}^\ell_U \to w$ if and only if G contains a $3k$-clique.*

Proof. We first recall the listing rules, for $xy \in \{\alpha\beta, \alpha\gamma, \beta\gamma\}$ and $i = 1, 2, \ldots, \ell - 1$, they are:

$$(1)\ \mathbf{S}^i_{xy} \to \mathbf{S}^*_{xy} \qquad (2_{xy})\ \mathbf{S}^i_{xy} \to \#\mathbf{N}^{i+1}_{xy}\$\mathbf{V}^{i+1}_{xy}\#$$
$$(3_{xy})\ \mathbf{N}^i_{xy} \to \#\mathbf{S}^{i+1}_{xy}\#\mathbf{V}^{i+1}_{xy}\$ \qquad (4)\quad \mathbf{N}^i_{xy} \to \sigma\mathbf{N}^i_{xy}\sigma$$

where $\sigma \in \{0, 1\}$, and the last assisting rule

$$(5)\mathbf{V}^i_{xy} \to \epsilon \mid \$\mathbf{W}'\$\mathbf{V}^{i+1}_{xy}$$

The proof in [3] proceeds by following the productions of the $\mathcal{G}$, and demonstrating that any resulting string must satisfy certain properties. Furthermore, if w is an encoding the a graph G as specified above, w will have these properties if and only if G has a $3k$-clique

We prove our extension of their theorem by showing that any string corresponding to the encoding of a graph that is accepted by the original grammar $\mathcal{G}$, can be produced using the listing productions (2_{xy}) and (3_{xy}), and the assisting production (5), at most ℓ times for some ℓ that we will later fix. Since these are the only non-linear productions which can be used more than once in any derivation, this will demonstrate for such an ℓ that the CFG $\mathcal{G}_U^\ell$ will accept an encoding w of a graph G if and only if G has a $3k$-clique.

Our proof follows that of [3], where we consider the sequence of productions which must be taken in order to derive w. We can only begin by the production $S \to w_1 a_{start} \mathbf{S}_{\alpha\gamma} c_{end} w_2$, where a_{start} appears in $CG(t_\alpha)$ for some $t_\alpha \in (C_k)$ and c_{start} appears in $CG(t_\gamma)$ for some $t_\gamma \in \mathcal{C}_k$. From here, we must derive $\mathbf{S}_{\alpha\gamma} \to CNG(t_\alpha)\mathbf{S}_{\alpha\gamma}CLG(t_\gamma)$ before exiting $\mathbf{S}_{\alpha\gamma}$ via the production $\mathbf{S}_{\alpha\gamma} \to \mathbf{S}^*_{\alpha\gamma}$, after which we can no longer return to $\mathbf{S}_{\alpha\gamma}$. The only way to produce the string $CNG(t_\alpha)\mathbf{S}_{\alpha\gamma}CLG(t_\gamma)$ is via the so-called listing productions $(2), (3), (4)$, thus we can confine our attention to them.

Note that $CNG(t_\alpha)$ consists of $k^2 \leq n^2$ binary encodings of vertices in G, whereas $CLG(t_\gamma)$ consists of $k^2 n \leq n^3$ such encodings. The only way to derive elements on the left of $\mathbf{S}^i_{\alpha\gamma}$ is by using the second listing production (2) and then deriving them via $\mathbf{N}^i_{\alpha\gamma}$ using (4). Repeated use of (4) allows for the derivation of exactly one of the binary encodings in $CNG(t_\alpha)$, and its corresponding reverse in $CLG(t_\gamma)$, say the sequence $\overline{v}$. Then each time we use the second production we are able to derive exactly one out of all k^2 sequences in CNG. By repeatedly applying (5), the nonterminal $\mathbf{V}^i_{\alpha\gamma}$ produced along with $\mathbf{N}^i_{\alpha\gamma}$ can derive all the binary sequences on the right side of $\overline{v}^R \in LG(u) \in CLG(t_\gamma)$, for some $u \in G$, and the $\mathbf{V}^{i+1}_{\alpha\gamma}$ derived from (3), after $\mathbf{N}^i_{\alpha\gamma}$ completes the derivation of $\overline{v}$, can construct all such binary sequences on the left side. There are at most n such sequences in $LG(u)$, thus we need use the production (5) at most n times to derive the rest of the terminals in $LG(u)$.

Thus each time we derive one of the $LG(u)$'s in $CLG(t_\gamma)$, during which we simultaneously derive one of the $NG(v)$'s in $CNG(t_\alpha)$, we use at most n non-linear productions, thus increasing the index of any nonterminal by at most n. Note in actuality, since (5) only involves $\mathbf{V}^i_{\alpha\gamma}$, we only increase the index of $\mathbf{V}^i_{\alpha\gamma}$ by this much; the index of $\mathbf{N}^i_{\alpha\gamma}$ and $\mathbf{S}_{\alpha\gamma}$ increase by at most two, since both of (2) and (3) are used at most once in this process, but for simplicity we will use n as the upper bound. Since this process must be repeated at most k^2 times, the total increase in the indices of the nonterminals is at most $nk^2 \leq n^3$ in the derivation of $CNG(t_\alpha)\mathbf{S}_{\alpha\gamma}CLG(t_\gamma)$

Now once we have produced the sentential form $CNG(t_\alpha)\mathbf{S}^i_{\alpha\gamma}CLG(t_\gamma)$, the only possibility is to "exit" via the production $\mathbf{S}^i_{\alpha\gamma} \to \mathbf{S}^*_{\alpha\gamma}$ for some i. From here, we must apply the production $\mathbf{S}^*_{\alpha\gamma} \to a_{mid}\mathbf{S}^1_{\alpha\beta}b_{mid}\mathbf{S}^1_{\beta\gamma}$, and then seek to derive

$$\mathbf{S}^1_{\alpha\beta} \to CNG(t_\alpha)\mathbf{S}^*_{\alpha\beta}CLG(t_\beta) \qquad \mathbf{S}^1_{\beta\gamma} \to CNG(t_\beta)\mathbf{S}^*_{\alpha\beta}CLG(t_\gamma)$$

Again, as just argued, both of these derivations can be completed using at most n^3 non-linear productions, and thus never producing a nonterminal of index greater than n^3. Once this has occurred, the rest of the string w can be derived via the exiting productions $\mathbf{S}^*_{\alpha\beta} \to a_{end}\mathbf{W}b_{start}$ and $\mathbf{S}^*_{\beta\gamma} \to b_{end}\mathbf{W}c_{start}$, as W can produce any string in Σ^*. Since no nonterminal with index greater than n^3 is ever produced, by setting $\ell = 2n^3$ it follows that our grammar $\mathcal{G}_U$ will accept the string w via the previous productions.

Now it is proven explicitly in [3], that if the derivation $\mathbf{S}_{xy} \to CNG(t)\mathbf{S}^*_{xy}CLG(t')$ can occur using the only the xy-listing rules, for any $t, t' \in \mathcal{C}_k$ and $xy \in \{\alpha\beta, \alpha\gamma, \beta\gamma\}$, then the

k-cliques t, t' must form a $2k$-clique $t \cup t'$. Since the set of all sentential forms, disregarding index, derivable from our grammar $\mathcal{G}_U^\ell$ is strictly a subset of its context free counterpart $\mathcal{G}$, this result immediately holds for $\mathcal{G}_U^\ell$ as well. Finally, since our derivation involved occurrences of all three of $\mathbf{S}_{\alpha\gamma} \to CNG(t_\alpha)\mathbf{S}_{\alpha\gamma}CLG(t_\gamma)$, $\mathbf{S}^1_{\alpha\beta} \to CNG(t_\alpha)\mathbf{S}^*_{\alpha\beta}CLG(t_\beta)$ and $\mathbf{S}^1_{\beta\gamma} \to CNG(t_\beta)\mathbf{S}^*_{\alpha\beta}CLG(t_\gamma)$, it follows that $t_\alpha \cup t_\beta \cup t_\gamma$ is a $3k$-clique.

The validity of the other direction can be demonstrated by following the derivations described above for any particular triple $t_\alpha, t_\beta, t_\gamma \in \mathcal{C}_k$ which together form a $3k$-clique, which completes the proof.

◀

Theorem 24. We have shown that for $\ell = 2n^3$, our grammar $\mathcal{G}_U^\ell$ accepts the string w iff $\mathcal{G}$ contains a $3k$-clique. The size of our grammar is then $|\mathcal{G}_U^\ell| = O(\ell) = O(n^3)$. Since the size of the string w encoding the graph G was $O(k^2n^{k+1})$, which can be constructed in $O(k^2n^{k+1}) < O(n^{k+3})$ time, it follows that if the membership of $w \in \mathcal{L}(\mathcal{G}_U^\ell)$ can be determined in time $O(|\mathcal{G}_U^\ell|^\alpha n^c)$, then the $3k$-clique problem can be solved in time $O(n^{c(k+3)+3\alpha})$, which proves the theorem. ◀

5.4 Metalinear and Superlinear Grammar Edit Distance

In this section we demonstrate a quadratic time algorithm for *metalinear* and *superlinear* grammars.

▶ **Definition 26** (k-metalinear)**.** G is said to be **k-metalinear** if every production is of the form:

$$S \to A_1 \dots A_t$$

$$A_i \to \alpha A_j \beta$$

Where $A_i \in Q \setminus \{S\}$, $\alpha, \beta \in \Sigma^*$, and $t \leq k$.

Thus, a k-metalinear language can have at most k linear nonterminals on the right hand side of a production. The metalinear languages (also referred as $LIN(k)$) strictly contain the linear languages. Furthermore, it has been shown that $Lin(k)$ is a strict subset if $Lin(k+1)$ for every $k \geq 1$, giving rise to a infinite hierarchy within the metalinear languages [27].

▶ **Definition 27** (superlinear)**.** G is said to be **superlinear** if there is a subset $Q_L \subset Q$ such that every nonterminal $A \in Q_L$ has only linear productions $A \to \alpha B$ or $A \to B\alpha$ where $B \in Q_L$ and $\alpha \in \Sigma$. If $X \in Q \setminus Q_L$, then X can have *non-linear productions* of the form $X \to AB$ where $A \in Q_L$ and $B \in Q$, or linear productions of the form $X \to \alpha A \mid A\alpha \mid \alpha$ for $A \in Q_L$, $\alpha \in \Sigma^*$. Superlinear grammars strictly contain the metalinear grammars.

Note that if we also allow both the nonterminals of the RHS to come from Q, then we get the entire class of context free grammars. A grammar G is superlinear iff every word $w \in L(G)$ can be expressed as the concatenation of words generated by linear grammars. This is a generalization of the metalinear languages, and can be thought of as the family $Lin(\infty)$. Superlinear grammars strictly contain the metalinear grammars, and are the regular closure of the linear languages. Several other nice properties of them have been well studied [27].

We now show how any metalinear grammar can be explicitly transformed into an equivalant superlinear grammar.

Conversion of Metalinear to Superlinear grammar. Let G_M be any k-linear grammar, we construct an equivalant superlinear grammar G_S. For every production of the form

$$S \to A_1 \dots A_t, \quad t \leq k$$

Add $t-1$ new nonterminals $A'_1, \dots A'_{t-1}$, and the following productions

$$S \to A'_{t-1} A_t$$

$$A'_i \to A'_{i-1} A_i \text{ for } i = 2, 3, \dots, t-1$$

$$A'_1 \to A_1$$

The result is a superlinear grammar G_S, with at most $p|P|$ new nonterminals, where p is the maximum number of nonterminals on the left hand side of any production. Under the assumption that $p = O(|P|)$, the $O(|P|n^2)$ time algorithm we present in this section for superlinear grammars gives an $O(|P|^2 n^2)$ algorithm for metalinear language edit distance.

Algorithm. We now present a quadratic time-complexity algorithm for computing the minimum edit distance to a superlinear grammar G. Let $\overline{x} = x_1 \dots x_n$ be our input string. The algorithm has two phases.

First Phase. In the first phase, we construct a graph $\mathcal{T}^R$, which is precisely the linear grammar edit distance graph $\mathcal{T}(Q_L, \overline{x})$ for the nonterminals in Q_L, but with the direction of every edge reversed and the weights kept the same. This, in effect, switches the roles of the source and sink vertices of $\mathcal{T}$. Computing the single source shortest path starting from t, by the symmetry of $\mathcal{T}$ and $\mathcal{T}^R$ we obtain the weight of the shortest path from $A^{i,j}$ to t in $\mathcal{T}$ for every nonterminal $A \in Q_L$ and $1 \leq i \leq j \leq n$. By the proof of the correctness of the linear language edit distance algorithm (Theorem 7), the weight of such a path is equal to the minimum edit distance of $x_i \dots x_j$ to a string s which can be legally produced starting from the state A. Thus computing single source shortest path from t in $\mathcal{T}^R$ allows us to construct a matrix $T_{i,j}(A) = c$ such that c is the minimum cost of deriving $x_i \dots x_j$ from A. This, as before, can be done in $O(n^2|P|)$ time.

Second Phase. Once we have $T_{i,j}(A)$ computed for all i, j and $A \in Q_L$, we begin the second phase where we construct a new graph $\mathcal{T}_{NL}$ with a new sink vertex t_{NL}, NL for non-linear, consisting of n clouds, each of which has a vertex for each of the non-linear nonterminals $Q \setminus Q_L$. We will denote the *ith* cloud by (i), and for any non-linear nonterminal $A_k \in Q \setminus Q_L$, we denote the vertex corresponding to A_k in (i) by A_k^i. Cloud (i) will then correspond to the substring $x_i x_{i+1} \dots x_n$, and for any nonterminal $A_k \in Q \setminus Q_L$, the weight of the shortest path from A_k^i to t_{NL} will be equal to the minimum edit distance between $x_i x_{i+1} \dots x_n$ to the set of strings legally derivable from A_k. Thus the vertex set of the graph is given by: $V(T_{NL}) = \{A_k^i \mid A_k \in Q \setminus Q_L,\ 1 \leq i \leq n\} \cup \{t_{NL}\}$. Let $null(A)$ denote the length of the shortest string legally derivable from A. We show how this can be computed for any CFG in $O(|Q||P| \log(|Q|))$ time in Theorem 30. We now describe the construction of the edges of $\mathcal{T}_{NL}$.

Construction of the edges.

1. For every non-linear production $A_k \to BC$, and each $1 \leq i \leq j < n$, create the edge $A_k^i \xrightarrow[T_{i,j}(B)]{BC} C^{j+1}$. *$B$ derives the substring $x_i x_{i+1} \dots x_j$ with a cost of $T_{i,j}(B)$*
2. For every non-linear production $A_k \to BC$ and each $1 \leq i \leq n$, create the edge $A_k^i \xrightarrow[null(B)]{BC} C^i$. *$B$ derives ϵ with a cost of $null(B)$*. Also create the edge $A_k^i \xrightarrow[T_{i,n}(B)+null(C)]{BC} t_{NL}$. *$B$ derives the substring $x_i x_{i+1} \dots x_n$ with a cost of $T_{i,n}(B)$, and C derives ε with a cost of $null(C)$*

3. For each production $A_k \to B$, and each $1 \le i \le n$, create the edge $A_k^i \xrightarrow[T_{i,n}(B)]{B} t_{NL}$. *$B$ derives the substring $x_i x_{i+1} ... x_n$ with a cost of $T_{i,n}(B)$*

▶ **Theorem 28.** *The weight W of the shortest path from S^1 to t_{NL} in $\mathcal{T}_{NL}$ is equal to the minimum language edit distance between $\overline{x}$ and G, and can be computed in $O(|P|n^2)$ time.*

Proof. The idea behind the proof is similar to that of the linear language edit distance algorithm. Every legal word $w \in L(G)$ can be derived starting from a string of nonterminals $A_1 A_2 \dots A_k$ (by suitable relabeling of the nonterminals) where $A_i \in Q_L$ for $1 \le i \le k$ and $S \to A_1 B_1 \to A_1 A_2 B_2 \to \cdots \to A_1 A_2 \dots A_k$ with $B_i \in Q \setminus Q_L$.

Let $w = \bigoplus_{i=1}^{k} w_i$ be a partition of the word such that w_i is the substring derived by A_i. Then if e_w is any sequence of editing procedures (deletions of a terminal in w, insertions of a terminal into w, or replacement of a terminal in w) which edits w into $\overline{x}$ given a specified set of legal production p_w which produce w, then we show how e_w can be partitioned into $e_{w_1}, e_{w_2}, \dots, e_{w_k}$, where e_{w_i} are the edits of e_w restricted to the substring w_i. This partition works as follows. Let ϵ be any single edit. If it is a deletion of a terminal in w_i, or the replacement of a terminal in w_i with another terminal, then we put ϵ in e_{w_i}. If ϵ is an insertion of a terminal between two terminals a and b which are both either in w_i, the result of replacement of a terminal in w_i, or the result of an insertion edit ϵ' in e_{w_i}, then we put ϵ in e_{w_i}. If the insertion is made on the boundary, say where either $a \in w_i$, a was the result of a replacement of a terminal and w_i, or a was an insertion made in e_{w_i}, and either $b \in w_{i+1}$, b was the result of a replacement of a terminal and w_{i+1}, or b was an insertion made in $e_{w_{i+1}}$, then we assign ϵ to e_{w_i} (we could just as easily assign to $e_{w_{i+1}}$, as long as the rule is consistent). In other words, we assign ϵ to the substring w_i of the lowest index of the two substrings corresponding to the terminals on either side of the insertion ϵ.

Now fix any such string w and set of edits e_w with corresponding partition $e_{w_1}, e_{w_2}, \dots, e_{w_k}$ such that e_w edits w into $\overline{x}$. Let $|e_{w_i}|$ be the total cost of the edits e_{w_i}, and let $\overline{x_i}$ be the result of applying e_{w_i} to w_i, then $\overline{x} = \bigoplus_{i=1}^{k} \overline{x_i}$. Now since the process of editing w_i into $\overline{x_i}$ is independent from the process of editing w_j into $\overline{x_j}$ for all $j \ne i$, the minimum edit distance from $\overline{x_i}$ to the set of strings that are legally derivable from A_i is less than or equal to the cost of e_{w_i} for all editing procedures e_{w_i}. By the proof of the linear grammar edit distance algorithm, for any $1 \le a \le b \le n$ and nonterminal A_i, the value $T_{a,b}(A_i)$ is equal to the minimum edit distance between $x_a \dots x_b$ and the set of strings legally derivable from A_i. Setting $l_i = \sum_{k=1}^{i-1} |\overline{x_k}|$, then we have in particular $T_{l_i+1, l_{i+1}}(A_i) \le |e_{w_i}|$. Furthermore, for any sequence $1 = \ell_1 \le \ell_2 \le \cdots \le \ell_{k+1} = n$, the path:

$$S^1 \xrightarrow[T_{\ell_1,\ell_2}(A_1)]{A_1 B_1} B_1 \xrightarrow[T_{\ell_2+1,\ell_3}(A_2)]{A_2 B_2} B_2 \longrightarrow \dots \longrightarrow t_{NL}$$

exists in $\mathcal{T}_{NL}$ with cost $\sum_{i=1}^{k} T_{\ell_i, \ell_{i+1}}(A_i)$. Setting $\ell_i = l_i$, it follows that $\sum_{i=1}^{k} T_{\ell_i,\ell_{i+1}}(A_i) \le |e_w|$ for any editing procedure which transforms a string w derivable from $A_1 \dots A_k$ into $\overline{x}$. In particular, this holds for the editing procedure with minimum cost, from which we conclude that W is at most the minimum language edit distance from $\overline{x}$ to G.

The fact that W can be no less than the minimum edit distance is easily seen, as every path corresponds to a derivation $S \to A_1 B_1 \to A_1 A_2 B_2 \to \cdots \to A_1 A_2 \dots A_k$, and a partition $\overline{x_1}, \overline{x_2}, \dots, \overline{x_k}$ such that $\overline{x} = \bigoplus_{i=1}^{k} \overline{x_i}$ and the cost of the path is the sum of the minimum costs of editing $\overline{x_i}$ into a string legally derivable from A_i over all $1 \le i \le k$. If W were less then the optimal, then the shortest path on $\mathcal{T}_{NL}$ would give a string of nonterminals $A_1 \dots A_k$ derivable from S such that $\overline{x}$ can be edited into a string legally derivable from

$A_1 \dots A_k$ with cost less than the language edit distance, a contradiction, which completes the proof.

Running Time. The first phase of the algorithm takes time $O(|P|n^2)$, as it entails running single source shortest path on the linear grammar edit distance graph $\mathcal{T}$. The graph $\mathcal{T}_{NL}$ constructed in the second phase has $O(|Q|n)$ vertices, and $O(|P_A|n)$ edges connecting to any vertex $A^i \in \mathcal{T}_{NL}$, where $P_A \subset P$ is the subset of productions with A on the left hand side. Thus the total number of edges is $O(|P|n^2)$, so running single source shortest path on a graph takes $O(|P|n^2)$ time, therefore the entire algorithm runs in $O(|P|n^2)$ time. ◀

▶ **Theorem 29.** *The language edit distance to any metalinear grammar can be computed in $O(|P|^2n^2)$ time.*

Proof. Follows directly from the above theorem and the conversion of any metalinear grammar to superlinear grammar. ◀

5.5 Proofs for the Linear Language Edit Distance Algorithm

▶ **Theorem 30.** *For any CFG $G = (Q, \Sigma, P, S)$, the set of values $\{null(A) \mid A \in Q\}$ can be computed in $O(|Q||P| \log(|Q|))$ time.*

Proof. The problem of finding $null(A)$ is solved by the algorithm given in [25]. Specifically, the problem of finding the shortest string derivable from A is application **(B)** of [25]. The algorithm takes as input a context free gramamr G and nonterminal $A \in Q$, and returns the value $null(A)$ in time $|P| \log(|Q|) + |P| = O(|P| \log(|Q|))$. Repeating the process for all nonterminals in $|Q|$ yields the desired runtime. ◀

Proof: Theorem 6. Let ρ be a sequence of legal productions which derives a string s from A_k, interspersed by d edits that edit s into $x_i \dots x_j$, where d is the optimal edit distance over the set of all such strings s derivable from A_k. We first show that the shortest path has weight d.

Base Case $d = 0$. If $d = 0$, then starting from the vertex $A_k^{i,j}$, we can follow edges for *legal productions* and *completing legal productions* according to ρ of weight 0 to reach t. (For example, if the first production in ρ is $A_k \to x_i A_{k_1}$, then the first edge taken will be $A_k^{i,j} \xrightarrow[0]{x_i} A_{k_1}^{i+1,j}$.)

Base Case $d = 1$. Let us now consider $d = 1$. The single error has caused by either substitution, or insertion, or deletion.

single substitution error. First, consider when a single substitution error has happened at position l. That is, if we had replaced x_l by some $a \in \Sigma$, the substring $s = x_i x_{i+1} ... x_{l-1} a x_{l+1} ... x_j$ can be derived from A_k with cost $d = 0$. Consider the series of legal productions made from A_k, up until the point where a is produced. At this point, there is some string $x_\ell \dots a$, or $a \dots x_{\ell'}$ which is left to be derived by some non-terminal $A_t \in Q$. WLOG, the string $x_\ell \dots a$ remains to be derived. For all the productions so far, we follow the edges created for *legal productions*, giving us a path from $A_k^{i,j}$ to $A_t^{\ell,l}$ of cost 0. At this point, we can take the replacement edge $A_t^{\ell,l} \xrightarrow[1]{x_l} A_q^{\ell,l-1}$, where $A_t \to A_q a$ is a production, or if $\ell = l$ then $A \to a$ is a production, so we take $A_t^{\ell,\ell} \xrightarrow[1]{x_l} t$. In the second case we are done. In the first we have the string $x_\ell \dots x_{l-1}$ left to derive from A_q, which can be done with cost 0 by again following the legal production edges, corresponding to the productions of ρ, to the sink. The concatenation of the paths from $A_k^{i,j}$ to $A_t^{\ell,l}$, of cost 0, then the edge of cost 1 to $A_q^{\ell,l-1}$, and then the path of cost 0 from $A_q^{\ell,l-1}$ to the sink, gives a full path from $A_k^{i,j}$ to t with a total cost of 1.

single deletion error. Now consider a single deletion error at position l. Hence, $s = x_i x_{i+1} ... x_{l-1} a x_l x_{l+1} ... x_j$ can be derived from A_k with $d = 0$. Then follow the series of legal productions of ρ until a is produced. At this point, we must either derive $x_\ell \dots x_{l-1} a$ or $a x_l \dots x_{\ell'}$ from a non-terminal A_t. WLOG $x_\ell \dots x_{l-1} a$ remains. Again, follow the legal edges from $A_k^{i,j}$ to $A_t^{\ell, x_{l-1}}$ with cost 0. Then take the edge $A_t^{\ell,l-1} \xrightarrow[1]{\epsilon(a)} A_q^{\ell,l-1}$, where $A_t \to A_q a$ is a production. Starting from $A_q^{\ell,l-1}$, we again follow the remaining legal edges, corresponding to the remaining productions in ρ which produce the rest of the string, to the sink. The whole path all together, then, takes us from $A_k^{i,j}$ to the sink with cost 1 as desired. One final case occurs if a is the last terminal derived in the sequence. Then either x_{l-1} or x_l was the last terminal derived when we are stopped, WLOG it is x_l with the production $A_r \to A_s x_l$. Then the last production of ρ must be a production $A_s \to a$. Then we have $null(A_s) = 1$. Then we can take legal edges from $A_k^{i,j}$ to $A_r^{l,l}$ with cost 0. We then take the edge $A_r^{l,l} \xrightarrow[null(A_s)]{x_l} t$, giving a full path to the sink with cost 1 as desired.

single insertion error. Now consider a single insertion error at position l. Hence, $s = x_i x_{i+1} ... x_{l-1} x_{l+1} ... x_j$ can be derived from A_k with $d = 0$. Again, consider the sequence of legal productions made until either x_{l-1} or x_{l+1} is derived, whichever happens first. WLOG x_{l-1} is derived first by a non-terminal $A_t \to x_{l-1} A_q$, with $x_{l+1} \dots x_\ell$ left to be derived from A_q. Then follow the corresponding legal production edges from $A_k^{i,j}$ to $A_q^{l,\ell}$, and then take the insertion edge $A_q^{l,\ell} \xrightarrow[1]{x_l} A_q^{l+1,\ell}$. From here, follow the edges given by the remaining legal productions of ρ, which takes us from $A_q^{l+1,\ell}$ to the sink with cost 0. Then the whole path has cost 1, as desired.

Induction. Assuming the result is true for errors up to $d-1$, the induction step for d edits is easy. Let s be the legally derivable string. Consider the sequence of legal productions in the production sequence of s up until either the production of a terminal which will be deleted, or a terminal which will be substituted, or to the point where a terminal will need to be inserted. let $x_i x_{i+1} \dots x_{r-1}$ and $x_{s+1} \dots x_j$ be the substrings that were derived by these legal productions so far at the point when we are stopped. Taking the corresponding legal production edges gives a path of cost 0 from $A_k^{p,q}$ to $A_t^{r,s}$ for some $A_t \in Q$. Now, in if the case is substitution, we apply the argument from the base case and arrive at a vertex $A_q^{r+1,s}$ or $A_q^{r,s-1}$ with cost 1, WLOG we are at $A_q^{r+1,s}$. Now there are $d-1$ remaining edits in between the string left to be legally derived by A_q and the string $x_{r+1} \dots x_s$. Thus the induction hypothesis applies, and we obtain a path of weight $d-1$ from $A_q^{r+1,s}$ to the sink. Concatenating the paths gives the desired path of length d.

In the case of deletion or substitution, we similarly follow the argument in the base case for $d = 1$, and then apply the induction hypothesis on the remaining substring left to be derived. The only distinct case to note is in the deletion case, if we are stopped at a point where we have derived all terminals in x except x_l, and there remains to be derived the substring $x_l a_1 a_2 \dots a_m$ of s – meaning that all of $a_1 \dots a_m$ must be deleted. In this case, we make the same argument at the beginning of the deletion base case, taking a path of cost 0 from $A_k^{i,j}$ to $A_r^{l,l}$. Suppose the next step in the derivation is $A_r \to x_l A_s$. Then we take the null edge $A_r^{l,l} \xrightarrow[null(A_s)]{x_l} t$ with cost at most m since $a_1 \dots a_m$ can be derived from A_s. Note that the cost must be exactly m, since the edit distance d is assumed to be optimal.

This completes all cases, thus the shortest path has weight at most d. Let d^* be the weight of the shortest path. Then reversing the process of reasoning taken above, any such path from $A_k^{i,j}$ to t of weight d^* gives rise to a production of s from A_k which can then be edited into $x_i \dots x_j$ using exactly d^* edits. Since d is the optimal edit distance, we have that $d^* = d$ is the weight of the shortest path as desired. Furthermore, considering the other

direction, this means that if d^* is the weight of the shortest path, then the minimum edit distance must also be d^*, which completes the proof.

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Proof: Theorem 7. The cost of the shortest path follows immediately from the previous theorem. Now there are $O(n^2)$ vertices in the graph for every nonterminal $A \in Q$. Hence, there are a total of $O(|Q|n^2)$ vertices. Let P_k denote the set of productions involving A_k on the left hand side. Then, for each $A_k^{i,j}$, the total out degree of that node is $O(|P_k|)$. Hence the total number of edges emanating from cloud (i,j) is $O(|P|)$, resulting in a total of $O(|P|n^2)$ edges. Since the maximum edge weight is bounded by 1, utilizing the best known single-source shortest path algorithm gives a $O(E(T)+V(T)) = O(|P|n^2)$[3] runtime algorithm to compute the weight of the shortest path from $S^{1,n}$ to t, which is the minimum edit distance from $\overline{x}$ to t.

◀

5.6 Normal Form for CFG's

▶ **Definition 31.** A context-free grammar G is in *normal form* if for any nonterminal $A \in Q$, all productions with A on the left hand side are of the form: (1) $A \to \beta B \mid B\beta \mid \beta$, or (2) $A \to CD$

It is well known that every CFG can be converted into Chompsky Normal Form. Since this normal form is strictly less restrictive than Chompsky normal form, it follows that every CFG can be converted into our normal form.

▶ **Lemma 32.** *Any k-ultralinear grammar can be converted into a k^*-ultralinear language in the above normal form, where $k^* \leq k\log(p)$, and p is the maximum number of nonterminals on the right hand side of any production.*

Proof. Consider any production $A \to A_1A_2\dots A_m$, where $A \in Q_t$ and A_j's are in partitions of lower index for $1 \leq j \leq m$. We add $\log(p)$ new partitions between Q_t and Q_{t-1}. We then make new nonterminals $A_1^1, A_2^1, \dots A_{\lceil m/2\rceil}^1$, and set the only production of each to be $A_i^1 \to A_{2i-1}A_{2i}$ for $i = 1, 2, ..., \lceil m/2\rceil$. If m is odd than $A_{\lceil m/2\rceil}^1 \to A_m$ will be the only production of the last nonterminal. We then repeat the process, creating nonterminals $A_1^2, \dots A_{\lceil m/4\rceil}^2$ and setting $A_i^2 \to A_{2i-1}^1 A_{2i}^1$. Finally, we create the production $A \to A_1^{\lceil\log(m)\rceil}A_2^{\lceil\log(m)\rceil}$. We place the terminals A_j^i in the partition that is depth $\lceil\log(m)\rceil - 1 + i$ lower than Q_t. Furthermore, for any production $A \to B$ where $A \in Q_t$ and $B \in Q_l$ with $l < t$, we can add a new non-termianl $B' \in Q_l$ such that its only production is $B' \to \epsilon$. We then change the production to $A \to BB'$. This does not increase the number of partitions. Doing the first process for all productions, the resulting grammar has at most $k\log(p)$ partitions, and after both processes at most $p|P|$ new nonterminals, since every production has at most p nonterminals on the right hand side. ◀

5.7 Proof of the CFG Exact Algorithm

Intuition + Sketch of the Algorithm:

Our algorithm makes crucial use of our earlier construction of the linear language edit distance graph $\mathcal{T}$. The essential idea is that, when tasked with deriving some substring $x_i \dots x_j$

[3] We assume $|P| \geq |Q|$, that is each nonterminal is involved in at least one production on the left.

from a nonterminal $A \in Q$ using a sequence of productions in P, and error productions corresponding to edit edges, there are two possibilities for the first production. Either the first production is a linear production, creating x_i or x_j with cost 0 if it is a legal production and cost 1 if it is an error production, or it is a non-linear production of the form $A \to CD$. In the latter case, no terminal is produced and we are now tasked with deriving $x(i:j) = x_i \dots x_j$ from CD. The first case is handled by the original construction of the graph $\mathcal{T}$. In the second case, C must derive some substring $x(i:\ell)$ and D must derive $x(\ell+1:j)$, each of which is a substring of size less than or equal to that of $x(i:j)$ (equal in the case that either one of C or D must derive all of $x(i:j)$, and the other derives no terminal). Here $i \le \ell \le j-1$ is referred to as the *splitting point*. To handle this situation, our algorithm computes shortest path on $\mathcal{T}$ in phases, where in each phase we compute shortest path to all substrings of a certain length, so that when computing the cost of the above non-linear production, we will have already computed the minimum cost of deriving $x(i:\ell)$ from C and $x(\ell+1:j)$ from D over all $i \le \ell \le j-1$.

Proof: Theorem 8. Note in this proof we work with the graphs $\overline{L_i}$ instead of $\overline{L_i^R}$ for simplicity. Since each path in one is just a reversal of the other, this is a trivial modification.

First, for any string $x \in \Sigma^*$, we write $x(p:q)$ to denote the substring $x_p x_{p+1} \dots x_q$. The proof is by induction on i. For $A^{\ell,\ell \in \overline{L}_1}$, consider the optimal editing procedure O from x_ℓ to the set of strings derivable from A. If this optimal edititing procedure does not involve a non-linear production, then the result follows from Theorem 6. If O does use a non-linear production, then it must be a production which nullifies a resulting non-terminal (Step. 1). This then corresponds to using a *null* edge created by our algorithm. Then necessarily O makes a series of non-linear productions, each time nullifying exactly one of the two resulting nonterminals, until we reach a nonterminal A_* from which we take a linear edge (possibly an error edge). The minimum cost of doing this is given by the *null* function, and thus following the corresponding null edges created in step 1 gives a path from $A^{\ell,\ell}$ to $A_*^{\ell,\ell}$. This cost of this path is precisely the optimal cost of nullifying all specified nonterminals. From A_* only linear productions are made, thus the cost of the shortest path from $A_*^{\ell,\ell}$ to t is precisely the cost of the remaining productions in O by Theorem 6.

Now assume the result for $1, \dots, i-1$, and fix any $A^{p,q} \in \overline{L}_i$, noting that necessarily $q-p+1 = i$, and consider an optimal series of legal and illegal (error) productions which produce $x(p:q)$ from A (note that every error production is a linear production). There are three cases, and consider the first production in this series. There are three cases:

If the first production is to derive x_p either via an insertion, replacement, or valid production, then this corresponds to a unique edge $A^{p,q} \to B^{p+1,q}$ with cost $\gamma \in \{0,1\}$, where γ depends on whether or not this production was an error. Suppose this edge takes us to $B^{p+1,q} \in \overline{L}_{i-1}$. In step 2, we create an edge from t to $A^{p,q}$ of cost $\gamma + T_{p+1,q}(B)$. By induction, $T_{p+1,q}(B)$ is the minimum edit distance between $x(p+1:q)$ and the set of strings which can be legally produced from B. Thus the cost of this edge is indeed the minimum edit distance between $x(p:q)$ and the set of strings which can be legally produced from A. The same argument holds when the terminal in question is x_q.

If the first production is a non-linear production $A \to BC$, and B and C each produce at least one terminal of $\overline{x}$, then it must be the case that there is some optimal splitting point $p \le \ell < q$ such that B derives $x(p:\ell)$ with cost c_1 and C derives $x(\ell+1,q)$ with cost c_2. Since each of these substrings is strictly smaller than i, they each correspond to a cloud in $\{\overline{L_1}, \dots, \overline{L_{i-1}}\}$, and since step 3 of the algorithm creates an edge with cost c which is at most $T_{p,\ell}(B) + T_{\ell+1,q}$ (since c is computed as minimum over all splitting points), by induction we know $c \le T_{p,\ell}(B) + T_{\ell+1,q} \le c_1 + c_2$. Since both c_1 and c_2 must necessarily be optimal costs

of deriving $x(p, \ell)$ from B and $x(\ell + 1, q)$ from C respectively, the cost of the edge created in step 3 is precisely $c_1 + c_2$. Thus taking this edge gives a shortest path which is indeed equal to the minimum edit distance between the substring $x_p \dots x_q$ and the set of strings which can be legally produced from A.

Finally, we consider the case that first production is a non-linear production $A \to BC$, and one of B or C creates no terminals in $\overline{x}$ (is nullified). WLOG, B is the nullified nonterminal. The corresponding edges are constructed in step 4, and takes us to $C^{p,q}$ with cost $null(A)$. By theorem 30, we can correctly compute $null(B)$ prior to commencement of the algorithm.

Now any edge taken from $A^{p,q}$ correspond to a derivation of $x(p : q)$ from A using both legal and error productions. Since the cost of these edges corresponds to the cost of the derivation, it must be the case that $T_{i,j}(A)$ is no less than the minimum edit distance between $x(p : q)$ and the set of strings which can be legally produced from A, which completes the proof.

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5.8 Proof of Theorem 14

Proof. If $\mathcal{A}$ returns c, then c is the length of the shortest path from $S^{1,n} \in \overline{L_n}$ to t in the graph $\mathcal{T}^{\mathcal{A}}$. Suppose there exists a $\mathbb{T} \in \mathcal{D}_{\mathcal{A}}$ with $\|\mathbb{T}\| < c$. Recall that $\|\mathbb{T}\|$ is the sum of the costs of all the nodes in $\mathbb{T}$. Thus, it suffices to show that the cost of the root of $\mathbb{T}$, plus the costs of all nodes rooted in the left and right subtrees of the root of $\mathbb{T}$, must be at least c. Our proof then proceeds inductively, working up from the leaves of $\mathbb{T}$. Let $[X_1, t, \omega_1], [X_2, t, \omega_1], \dots, [X_k, t, \omega_k]$ be the leaves of $\mathbb{T}$. Since each of the $[X_i, t, \omega_i]$'s are leaves, each of these paths must use only the linear edges from the original linear grammar edit distance graph $\mathcal{T}$ – so these edges must also exist in $\mathcal{T}^{\mathcal{A}}$. Thus for $1 \leq i \leq k$, the shortest path from X_i to t in $\mathcal{T}^{\mathcal{A}}$ is at most ω_i.

Now let $[A_\star^{p,q}, B_\star^{r,s}, \omega_\star]$ be any non-leaf node in $\mathbb{T}$, with left and right children $[A_L^{r,\ell}, \cdot, \omega_L]$ and $[A_R^{\ell+1,s}, \cdot, \omega_R]$ respectively. Let $\overline{\omega_L}$ and $\overline{\omega_R}$ be the sum of the costs of all nodes in the subtree rooted at $[A_L^{r,\ell}, \cdot, \omega_L]$ and $[A_R^{\ell+1,s}, \cdot, \omega_R]$, respectively. Note that because any node is included in the subtree rooted at itself, we include ω_L in the value $\overline{\omega_L}$ and ω_R is in the value $\overline{\omega_R}$.

Now suppose that the weight of the shortest path from $A_R^{\ell+1,s}$ to t and from $A_L^{r,\ell}$ to t in $\mathcal{T}^{\mathcal{A}}$ is at most $\overline{\omega_R}$ and $\overline{\omega_L}$ respectively. We would like to show that the shortest path from $A_\star^{p,q}$ to t in $\mathcal{T}^{\mathcal{A}}$ is at most $\omega_\star + \overline{\omega_R} + \overline{\omega_L}$.

Now since $[A_\star^{p,q}, B_\star^{r,s}, \omega_\star]$ ends in cloud (r, s), by property 1 of PET's in $\mathcal{D}_{\mathcal{A}}$, it must be the case that $\mathcal{A}$ computes non-linear edges for the cloud (r, s). From $B_\star^{r,s}$, a non-linear edge e, corresponding to the production $B_\star \to A_L A_R$, is taken with splitting point ℓ. By property 2 of trees in $\mathcal{D}_{\mathcal{A}}$, the splitting point ℓ must have been considered by $\mathcal{A}$ when computing the cost of this edge. Thus, the cost of the edge e in $\mathcal{T}^{\mathcal{A}}$ is at most the cost of the shortest path from A_L to t plus the cost of the shortest path from A_R to t. By the inductive hypothesis, the cost of e is then at most $\overline{\omega_R} + \overline{\omega_L}$. Since $\omega_\star$ is the cost of a path of consisting only of linear edges from $A_\star^{p,q}$ to $B_\star^{r,s}$, this path must also exists in $\mathcal{T}^{\mathcal{A}}$. Thus following this path from $A_\star^{p,q}$ to $B_\star^{r,s}$ and then taking e results in a path that exists in $\mathcal{T}^{\mathcal{A}}$, going from $A_\star^{p,q}$ to t, with cost at most $\omega_\star + \overline{\omega_R} + \overline{\omega_L}$, which is the desired result.

Finally, note that since $\mathcal{A}$ creates all *null* edges of the graph created by the exact algorithm $\mathcal{T}^{OPT}$, the above result holds in the case where one of the left or right children of $[A_\star^{p,q}, B_\star^{r,s}, \omega_\star]$ is a nullified node, since then the cost of the *null* edge is just the cost of nullifying the specified nonterminal, and the inductive hypothesis holds for the other, non-nullified, child. This completes all cases. Using this argument inductively, it follows that

c must be no greater than the cost of the root of $\mathbb{T}$ plus the costs of all nodes rooted in the left and right subtrees of the root of $\mathbb{T}$, a contradiction, which completes the proof.

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